Coalgebraic Semantics for Intuitionistic Modal Logic

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Coalgebras

Definition

Given a category $\mathbb{C}$, and an endofunctor $F : \mathbb{C} \to \mathbb{C}$, a pair (A, f) of an object A and a morphism $f : A \to F(A)$ is called an (F-)*coalgebra*.

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Given two coalgebras (A, f_A) and (B, f_B), $h : A \to B$ is a *coalgebra morphism* between (A, f_A) and (B, f_B) if the following commutes:

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & B \\ {\scriptstyle f_A}\big\downarrow & & \big\downarrow{\scriptstyle f_B} \\ F(A) & \xrightarrow[\;F(h)\;]{} & F(B) \end{array}$$

Figure 1: Coalgebra morphism compatibility

We write $\mathbf{CoAlg}(F)$ for the category of F-coalgebras and coalgebra morphisms.

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Definition

Let X be a Stone space. $\mathcal{V}(X)$ is the set of all *closed* subsets of X, with a subbase (*hit-or-miss topology*) given by:

$$[U] = \{C \in \mathcal{V}(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in \mathcal{V}(X) : C \cap V \neq \emptyset\},$$

where U, V are clopen in X.

Positive Modal Logic

If we restrict to the positive fragment of modal logic, something similar; look at the language:

$$\mathcal{L} = \wedge \mid \vee \mid \Box \mid \top \mid \bot.$$

The positive fragment of K can be axiomatised over positive logic by adding the axioms $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box\top = \top$.

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This can be interpreted over positive Kripke frames: triples $(P, \leq, R)$ where:

1. $(P, \leq)$ is a poset;
2. (P, R) is a Kripke frame;
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Topologize: Priestley spaces with a closed relation R satisfying the compatibility condition.

Just like in the classical case, there are analogous constructions providing coalgebraic semantics:

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1. *Given a poset* $(P, \leq)$ *we can consider* $(\mathrm{Up}(P), \supseteq)$*; this is an endofunctor, and coalgebras for this functor are precisely the positive Kripke frames;*

Coalgebraic Semantics of Positive Modal Logic

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2. *Given a Priestley space* $(X, \leq)$ *we can associate to it* $\mathcal{V}_{\uparrow}(X)$*, the set of closed upsets, with the hit-or-miss topology on clopen upsets; this is an endofunctor, and coalgebras for this functor are precisely the Priestley spaces with a closed relation satisfying the compatibility.*

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Proof.
Given a Priestley space with a closed relation $(X, \leq, R)$, we can consider the coalgebra:

$$p_R : X \to \mathcal{V}_{\uparrow}(X)$$
$$x \mapsto R[x].$$

Conversely, if $p : X \to \mathcal{V}_{\uparrow}(X)$, define the relation:

$$xRy \iff y \in p(x).$$

Similar in the case of the powerset. □

In some settings it is natural to want to strengthen positive to intuitionistic modal logic. Consider the following language:

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By topologizing it, to obtain a complete semantics for the intuitionistic part, we need to consider Esakia spaces with a closed relation R satisfying the compatibility $R = \leq \circ R \circ \leq$.

A Coalgebraic Puzzle

A natural conjecture: the coalgebras for Up over the category of Posets with $\leq$-bounded morphisms are the intuitionistic Kripke frames; the coalgebras for $\mathcal{V}_{\uparrow}$ over the category of Esakia spaces are the Esakia spaces with a closed relation.

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Problem (Litak, 2014): If we are in the category of Esakia spaces, we must have the coalgebra map

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Our contribution: We propose a solution to this in specific cases by working with a different endofunctor.

From Priestley to Esakia

Intuitionistic Lifting

When it comes to Esakia spaces, our key technical tool lies in the following:

Theorem (A., 2024)
The inclusion $I : \mathsf{Esa} \to \mathsf{Pries}$ *of Esakia spaces in Priestley spaces admits a right adjoint* V_G.

The construction of V_G works through a step-by-step construction. It is dual to the free Heyting algebra generated by a distributive lattice, and the step-by-step approach layers infinitely many implications.

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Given X a Priestley space, the elements of $V_G(X)$ have the form:

$$(x, C_0, C_1, ..., C_n, ...)$$

where C_0 is a closed and rooted subset of X, C_1 is a rooted and further special subset of $V(X)$, and so on.

What about the case of Kripke frames? Here we make an additional restriction:

Definition

Let $(X, \leq)$ be a poset. We say that X is *image-finite* if for each x, the set $\uparrow x = \{y : x \leq y\}$ is finite.
ImFinPos is the category of image-finite posets with $\leq$-bounded morphisms.

Intuitionistic Lifting (cont.d)

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ImFinPos is the category of image-finite posets with $\leq$-bounded morphisms.

Then we have the following:

Theorem (A., 2024)

The inclusion I : ImFinPos $\to$ Pos *admits a right adjoint* P_G.

Given $(X, \leq)$, the elements of $P_G(X)$ are sequences:

$$(x, C_0, C_1, ..., C_n, ...).$$

where C_i are rooted, finite subsets.

Coalgebras for Intuitionistic Modal Logic

A special consequence of the above result is the following:

Proposition

Let X be an Esakia space, Y a Priestley space, and assume that $f : X \to Y$ is a Priestley morphism. Then there is a unique Esakia morphism $\bar{f} : X \to V_G(Y)$, extending f.

Theorem (*Main Result on Esakia Spaces*)
The functor $V_G(\mathcal{V}_{\uparrow}(-))$ *provides a coalgebraic representation of Esakia spaces with a closed relation.*

Lifting Positive to Intuitionistic Representations

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Proof.
Given an Esakia space $(X, \leq, R)$, consider its positive coalgebra representation:

$$p_R : X \to \mathcal{V}_\uparrow(X)$$

Using the above result, this lifts to:

$$\tilde{p_R} : X \to V_G(\mathcal{V}(X)).$$

Conversely, if $g : X \to V_G(\mathcal{V}_\uparrow(X))$ is a coalgebra map, then $\pi_0 g : X \to V_\uparrow(X)$ is a coalgebra, which corresponds to an Esakia space with a closed relation. □

Similarly, we have:

Theorem (*Main Result on Image-Finite Posets*)
The functor $P_G(\mathrm{Up}(-))$ *provides a coalgebraic representation of Image-Finite intuitionistic Kripke frames.*

Consequences and Further Remarks

We have the following consequences of the main results:

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In a recent preprint, it is shown that $\mathbf{Pos}_p$ is not complete; hence the inclusion $\mathbf{Pos}_p \to \mathbf{Pos}$ cannot have a right adjoint (since these preserve limits). So in this sense our result is "best possible".

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An *intuitionistic neighbourhood frame* is a triple $(X, \leq, N)$ of a poset together with a monotone map $N : X \to \mathcal{P}(\mathrm{Up}(X))$, where $\mathcal{P}(-)$ is ordered by inclusion. The morphisms between such frames are functions $f : X \to X'$ satisfying

$$a' \in N'(f(x)) \iff f^{-1}(a') \in N(x)$$

for all $x \in X$ and $a' \subseteq X'$. We denote by ImFinN the category of image-finite neighbourhood $\Box$-frames.

Theorem

There is an equivalence between $\mathsf{CoAlg}(\mathrm{P}_G(\mathcal{P}(\mathrm{Up}(-))))$ *and the category* ImFinN.

Definition

Let F : **Pries** $\to$ **Pries** be an endofunctor on the category of Priestley spaces; we define F^* : **Esa** $\to$ **Esa**: the *intuitionistic lifting* of F to be the functor obtained by composition in the following diagram:

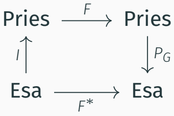

Figure 2: Intuitionistic Lifting of functor F

Intuitionistic Liftings in general

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The results presented so far indicate a way to move from *positive distributive logics* to *intuitionistic logics*, using the mechanism of intuitionistic lifting.

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The results presented so far indicate a way to move from *positive distributive logics* to *intuitionistic logics*, using the mechanism of intuitionistic lifting.

We leave a systematic study of the properties of intuitionistic lifting of functors, in a coalgebraic setting, for further work.

Thank you!

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