Unification with Simple Variable Restrictions and Admissibility of Pi2-Rules

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Non-Standard Rules

Motivation

Non-standard Π_2-rules have been used often for axiomatisation: examples include Gabbay's rule:

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The rules of the Strict Implication Calculus:

$$\frac{\forall p\,((p \rightsquigarrow p) \wedge (\phi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi)) \to \chi}{(\phi \rightsquigarrow \psi) \to \chi.}$$

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Warning: Throughout the universal quantifiers are not part of our language; they signal which variable is forbidden from appearing elsewhere in the formulas.

Admissibility of Pi2-Rules

Main applications: axiomatisations, proving their admissibility. Recent work by Bezhanishvili et.al (2023) developed a systematic method for recognising some of these rules, with syntactic specifications.

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This syntactic specification is forcibly classical, and hence left out several interesting kinds of rules, such as Takeuti-Titani rule.

In recent work (A., 2024, submitted), I developed a framework to discuss Π_2-rules, where they are shown to generalize usual rules. This raised the question of how one can recognise the admissibility of such rules.

Pi2-Rules in General

We work with an algebraizable logic $\vdash$ in a language $\mathcal{L}$, where its equivalent algebraic semantics K is a variety. We also fix a Hilbert style calculus for $\vdash_{\mathsf{L}}$.

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Definition

Let $\Gamma = \{\phi_i(\overline{p}, \overline{q}) : i \leq n\}$ and $\psi(\overline{q})$ be formulas of $\mathcal{L}$. The Π_2-rule associated with this sequence of formulas is denoted $\forall\overline{p}\Gamma/^2\psi$ and displayed as:

$$\frac{\forall\overline{p}\ (\phi_0(\overline{p}, \overline{q}), ..., \phi_n(\overline{p}, \overline{q}))}{\psi(\overline{q}).}$$

We write $F_c(\Gamma) = \{\overline{p}\}$ for the bound context.

Whenever the bound context is empty, the rule is referred to as a *standard* or Π_1-rule.

Definition

Let Σ be a set of Π_2-rules. Given a formula ϕ we say that ϕ is *derivable* using the Π_2-rules in Σ, and write $\vdash_{L\oplus\Sigma} \phi$, provided there is a sequence $\psi_0, \dots, \psi_n$ of formulas such that:

- $\psi_n = \phi$;
- For each ψ_i we have that either:
    1. ψ_i is an instance of an axiom of $\vdash_L$ or,
    2. ψ_i is obtained using a rule from $\vdash_L$, from some previous $\psi_{j_0}, \dots, \psi_{j_k}$ or,
    3. $\psi_i = \chi(\overline{\xi}/\overline{q})$ and $\psi_{j_k} = \mu_k(\overline{r}/\overline{p}, \overline{\xi}/\overline{q})$ for $0 \leq j_k < i \leq n$, where
        - $\overline{r}$ is a renaming of $\overline{p}$, away from $\overline{\xi}$, i.e., a set of fresh variables not ocurring in $\overline{\xi}$;
        - $\Delta = \{\mu_k(\overline{p}, \overline{q}) : k \in \{0, \dots, m\}\}$;
        - $\forall\overline{p}\Delta/^2\chi \in \Sigma$;
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Admissibility

Admissibility in Π_2-rules

Definition
Let $\Gamma/^2\phi$ be a Π_2-rule. We say that the rule $\Gamma/^2\phi$ is *admissible* in $\vdash_{\mathsf{L}}$ if for all ψ:

$$\vdash_{\mathsf{L}\oplus\Gamma/^2\phi} \psi \implies \vdash_{\mathsf{L}} \psi.$$

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Remark: If $\forall \overline{p}\Gamma/^2\phi$ is a Π_2-rule and $C = F_C(\Gamma)$, then $\forall \overline{p}\Gamma/^2\phi$ is admissible over $\vdash_{\mathsf{L}}$ if and only if whenever σ is a C-invariant substitution and we have $\vdash_{\mathsf{L}} \sigma(\Gamma)$, then we have also $\vdash_{\mathsf{L}} \sigma(\phi)$.

Admissibility in Π_2-rules

Given a logic $\vdash$, by the $\vdash$-*admissibility problem for* Π_2-*rules* we mean the problem of determining, given a triple $(\Gamma, \phi, \overline{p})$, whether the Π_2-rule $\forall \overline{p} \Gamma /^2 \phi$ is admissible over $\vdash_{\mathsf{L}}$.

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The $\vdash$-admissibility problem for Π_1-rules need not be decidable. It is decidable for a substantial number of logical systems encountered in practice:

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2. IPC;
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Our contribution is identifying a number of sufficient conditions to reduce the problem of Π_2-rule admissibility to that of Π_1-rule admissibility.

Definition

We say that a logic $\vdash$ has the Maehara interpolation property if for any finite sets of formulas $\Sigma(\overline{p}, \overline{q}), \Delta(\overline{q}, \overline{r}), \Sigma'(\overline{p}, \overline{q})$ in the language $\mathcal{L}$ the following holds:

$$\Sigma(\overline{p}, \overline{q}) \cup \Delta(\overline{q}, \overline{r}) \vdash \Sigma'(\overline{p}, \overline{q}) \implies \exists \Pi(\overline{q}) \text{ such that } \Delta(\overline{q}, \overline{r}) \vdash \Pi(\overline{q})$$
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Main Result on Admissibility

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We say that a logic $\vdash$ has right uniform deductive interpolation if for any finite set of formulas $\Sigma(\overline{p},\overline{q})$ there exists a finite set of formulas $\Pi(\overline{q})$ such that:for any finite set of formulas $\Delta(\overline{q},\overline{r})$

$$\Sigma(\overline{p},\overline{q}) \vdash \Delta(\overline{q},\overline{r}) \iff \Pi(\overline{q}) \vdash \Delta(\overline{q},\overline{r}).$$

Definition

We say that $\vdash$ has left-finitary uniform deductive interpolation if for any finite set of formulas $\Delta(\overline{q}, \overline{r})$ there is a finite collection of finite sets of formulas $\Theta_1(\overline{q}), \ldots, \Theta_n(\overline{q})$ such that:

(i) $\Theta_i(\overline{q}) \vdash \Delta(\overline{q}, \overline{r})$ for each $i \leq n$ and

(ii) for any $\overline{p}$ and $\Sigma(\overline{p}, \overline{q})$,

$$\Sigma(\overline{p}, \overline{q}) \vdash \Delta(\overline{q}, \overline{r}) \implies \Sigma(\overline{p}, \overline{q}) \vdash \Theta_i(\overline{q})$$

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for some $i \leq n$.

When there is a single Θ_i this is referred to as left-uniform deductive interpolation

Theorem

Suppose that $\vdash$ has

1. *the Maehara Interpolation Property;*
2. *Right-Uniform Deductive Interpolation;*
3. *Left-Finitary Deductive Uniform Interpolation.*

Then if both $\vdash$ itself and the $\vdash$-admissibility problem for standard rules are decidable, so is the $\vdash$-admissibility problem for Π_2-rules.

Main Result (cont.d)

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Many examples coming from logics with uniform interpolation: S5, GL, Grz, IPC, LC and lax logic.

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Many examples coming from logics with uniform interpolation: S5, GL, Grz, IPC, LC and lax logic.

Further examples given by $(\wedge, \top, \to)$-fragment of IPC, and the $(\ell, \wedge, \top, \to)$-fragment of Lax Logic.

Unification with Simple Variable Restrictions

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Definition

Given an equational theory E, and a finite set of variables C, an *E-unification problem with simple variable restriction* (briefly a *C-unification problem*) is a finite set of pairs of terms in the variables $\overline{p}$ (with $C \subseteq \overline{p}$):

$$(P_C) \qquad (\phi_1(\overline{p}), \psi_1(\overline{p})), \ldots, (\phi_k(\overline{p}), \psi_k(\overline{p}));$$

a solution to such a problem or a *C-unifier* is a C-invariant substitution σ of domain $Fm_{\mathcal{L}}(\overline{p})$ such that

$$\sigma(\phi_1) =_E \sigma(\psi_1), \ldots, \sigma(\phi_k) =_E \sigma(\psi_k).$$

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A C-unification basis from this set is a subset $B \subseteq U_E^{svr}(P_C)$ such that for every $\sigma' \in U_E^{svr}(P_C)$ there is $\sigma \in B$ such that $\sigma' \leq_C \sigma$ holds; a most general C-unifier (C-mgu) of (P_C) is a $\sigma \in U_E^{svr}(P_C)$ such that $\{\sigma\}$ is a C-unification basis.

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Definition

We say that E has *finitary simple-variable-restriction (svr) unification type* iff every C-unification problem (P_C) has a finite C-unification basis; E has *unitary scr-unification type* iff every C-unification problem (P_C) has a C-mgu.

Proposition

Assume that $\vdash_S$ is decidable. Then if E has finitary svr-unification type (with computable finite C-unification bases), then the $\vdash$-admissibility problem for Π_2-rules is decidable too.

Algebraic Representation of the Problem

Given a finitely presented E-algebra $\mathcal{A}$, we write it as $\mathcal{A}^*$ when we see it in the opposite category E (thus, $\mathcal{A}^*$ is just a formal dual of $\mathcal{A}$); in particular $\mathsf{F}(X)^*$ is the formal dual of $\mathsf{F}(X)$, the free algebra on the finitely many generators X. Given an object $\mathcal{B}^*$ in E, we write $\mathrm{Sub}_r(\mathcal{B}^*)$ for the set of regular subobjects.

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An E-unification problem with simple variable restrictions is a pair $(\mathcal{A}, C = \overline{p})$, such that $\mathcal{A}^* \in \mathrm{Sub}_r(\mathsf{F}(X)^* \times \mathsf{F}(C)^*)$. A solution is a homomorphism $\sigma : \mathsf{F}(X) \to \mathsf{F}(Z)$, such that $\sigma^* \times 1$ factors such that Figure 1 commutes:

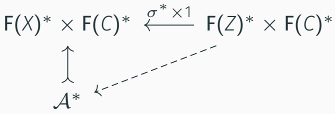

Figure 1: Solution to Unification Problem with Simple Variable Restrictions

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$$\begin{array}{ccc} \mathsf{F}(X)^* \times \mathsf{F}(C)^* & \xleftarrow{\sigma^* \times 1} & \mathsf{F}(Z)^* \times \mathsf{F}(C)^* \\ \uparrow & \swarrow & \\ \mathcal{A}^* & & \end{array}$$

Figure 1: Solution to Unification Problem with Simple Variable Restrictions

This ‘algebraic’ approach to C-unification is equivalent to the ‘symbolic’ approach.

Equivalents of the interpolation properties

The interpolation properties introduced have algebraic equivalents:

1. Maehara interpolation property $\rightsquigarrow$ “Injections are Transferable”;
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Definition

Let C be a category with finite limits. Given $T \in \mathrm{Sub}_r(X \times Z)$, we say that a finite collection $B_1, \ldots, B_n \in \mathrm{Sub}_r(Z)$ is a $\forall_X$-*factorization* of T if:

1. $\pi_Z^{-1}(B_i) \leq T$ for each $i \leq n$;
2. for every $C \leq Z$, such that $\pi_Z^{-1}(C) \leq T$, there is some $i \leq n$ such that $C \leq B_i$.

We say that C has the $\forall$-*factorization property* if for all objects X, Z and any $S \in \mathrm{Sub}_r(X \times Z)$, there is a $\forall_X$-factorization of S.

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Proposition

The logic $\vdash$ has left-finitary deductive interpolation if and only if $\mathrm{Alg}_{fp}^{op}(E)$ has the $\forall$-factorization property.

Outline of the Proof

Theorem

Suppose that K *satisfies (IT), Coherence and* $\forall$*-factorization property. Then:*

1. *If E has finitary unification type, then it has finitary unification type for the problem with simple variable restrictions.*
2. *If E has unitary unification type and uniform interpolation, then E has unitary type for the problem with simple variable restrictions.*
3. *If E-unification is decidable and* $\forall$*-factorizations are computable, then E-unification with simple constant restrictions is decidable as well.*

Proof.
(Idea) We show that in these conditions, a basis of C-unifiers can be constructed from a larger base of unifiers.

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Proof.
(Idea) We show that in these conditions, a basis of C-unifiers can be constructed from a larger base of unifiers.
We then need to show that substitutions still commute with finitary ∀-decompositions – a generalization of the Beck-Chevalley property for these collections (this is the main technical fact necessary for the result). □

Examples

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Our method allows us to show the admissibility of the Takeuti-Titani rule using bisimulation semantics. In fact, under this perspective, its admissibility is trivial: the uniform interpolant of the antecedent is the consequent.

We can also use our method to disprove uniform interpolation: in the theory of implicative semilattices the problem

$$((x \to z) \wedge (y \to z) \to z, \top)$$

has two incomparable C-unifiers. If it had uniform interpolation, this would be impossible.

Examples

A key example is the case of $(\wedge, \to, \ell)$-fragment of Lax logic. One of our contributions in this paper is to give a proof that this logic is in the conditions of the theorem.

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Namely, we show that such subalgebras of finite projective algebras of this system are again projective.

Open questions

The present work leaves open several interesting lines of questioning:

1. Simple variable restrictions is an instance of the more complex linear variable restriction problems. It would be interesting to understand how stronger rules relate to such unification problems. This can have applications in positive model theory.

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1. Simple variable restrictions is an instance of the more complex linear variable restriction problems. It would be interesting to understand how stronger rules relate to such unification problems. This can have applications in positive model theory.
2. Some well-known examples of logics with decidable admissibility problems fall outside of the scope of the present analysis – notably the system S4. It would be interesting to study the simple variable unification type of this logic, and the associated admissibility problem.

Thank you!