Colimits of Heyting Algebras through Esakia Duality

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February 12, 2024

Abstract

In this note we generalize the construction, due to Ghilardi, of the free Heyting algebra generated by a finite distributive lattice, to the case of arbitrary distributive lattices. Categorically, this provides an explicit construction of a left adjoint to the inclusion of Heyting algebras in the category of distributive lattices This is shown to have several applications, both old and new, in the study of Heyting algebras: (1) it allows a more concrete description of colimits of Heyting algebras, as well as, via duality theory, limits of Esakia spaces, by knowing their description over distributive lattices and Priestley spaces; (2) It allows a direct proof of the amalgamation property for Heyting algebras, and of related facts; (3) it allows a proof of the fact that the category of Heyting algebras is co-distributive. We also study some generalizations and variations of this construction to different settings. First, we analyse some subvarieties of Heyting algebras – such as Boolean algebras, KC and LC algebras, and show how the construction can be adapted to this setting. Second, we study the relationship between the category of image-finite posets with p-morphisms and the category of posets with monotone maps, showing that a variation of the above ideas provides us with an appropriate general idea.

1 Introduction

One of the many attractive features of duality theory as it pertains to the theory of Boolean algebras and distributive lattices is that many constructions which are quite generic in algebraic terms (e.g., word constructions), become more concrete when put in dual terms. A prime example of this lies in the coproduct of Boolean algebras: in contrast to the abstract description of coproducts of algebras, given Boolean algebras $\mathcal{B}_0$ and $\mathcal{B}_1$, their coproduct $\mathcal{B}_0 \oplus \mathcal{B}_1$ can be described as

$$\mathsf{Clop}(X_0 \times X_1),$$

where $\mathcal{B}_0^* = X_0$ and $\mathcal{B}_1^* = X_1$ are the Stone duals of these algebras. This makes it possible to prove several general facts by simple observations – for instance, the fact that the canonical coprojections into a coproduct are injective follows from the fact that the projections of Stone spaces onto its factors are surjective. Similar facts hold regarding distributive lattices and their dual Priestley spaces.

At the root of this simplicity in describing coproducts, lies the fact that free algebras in these varieties admit rather explicit constructions. Indeed, the structure of free Boolean algebras and free distributive lattices on any number of generators is well-understood, and has been extensively exploited both for algebraic and set-theoretic purposes (see [21, Chapter 9-10] for an extensive discussion).

In sharp contrast to this, the structure of free Heyting algebras is much more difficult: already the free algebra on one generator is infinite, and the free algebra on two generators is notoriously difficult. As such, there has been considerable attention devoted to such objects. Notably, the construction of the n-universal model (due independently to Bellissima [3], Grigolia [18]) and Shehtman [27], allowed a more concrete description of the completely join-irreducible part of the free n-generated Heyting algebras; whilst the construction of the free n-generated Heyting algebra by Ghilardi [16] (which generalized previous work by Urquhart [29], and was in turn generalized for all finitely presented Heyting algebras by Butz [9]) and the associated ideas allowed the development of semantic proofs of several key facts regarding this category of algebras: the fact that free Heyting algebras are bi-Heyting algebras, the fact that the first-order theory of Heyting algebras has a model completion (due in its syntactic form to [24], and proven as such by Ghilardi and Zawadowski [17]), the fixed-point elimination theorem (proved in its syntactic form by Ruitenburg [23] and in its semantic form by Ghilardi and Santocanale [26]), amongst others. All of these make explicit and non-trivial use of the fact that the algebras of interest are finitely presented, and exploit the finitarity to work with discrete dualities at key steps. As such, a description of general free constructions of Heyting algebra seems to not have been addressed in the current literature.

From our point of view, the main hurdle in this is realizing the crucial point of finiteness: the above constructions by Ghilardi rely heavily on the properties of Birkhoff duality for finite distributive lattices, and it is not obvious how to generalize such facts to the infinite case. The purpose of this paper is to provide such a generalization. In place of Birkhoff duality we use Priestley duality, and show that this allows us to construct all free Heyting algebras in a way which generalizes existing work. This amounts essentially to describing a left adjoint to the inclusion of **HA**, the category of Heyting algebras, in **DL**, the category of distributive lattices.

Having such a description at hand, we then proceed to employ it to provide some results on the structure of Heyting algebras, some known and some new:

1. A description of free Heyting algebras on any number of generators is given;
2. A description of coproducts of Heyting algebras is given, and it is shown that the category of Heyting algebras is co-distributive;
3. A description of pushouts of Heyting algebras is given, and it is shown directly that the coprojections of Heyting algebras to the pushout are injective (yielding, as a corollary, the amalgamation property).

We then proceed to extend these results this in several directions, remaining in the realm of Heyting algebras and associated categories. We first show that this construction can be adapted when dealing with some subvarieties of Heyting algebras. Rather than providing a general theory, we focus on two important and illustrative cases: KC algebras and LC algebras. Our analysis of the latter also reveals a surprising property of Gödel logic which we were not aware before: that every formula in Gödel logic is equivalent to one of modal depth 2. Finally, we look at the relationship between the category of posets with p-morphisms and the category of posets with monotone maps, and provide a left adjoint to the inclusion which is heavily inspired by the above constructions. We briefly discuss some coalgebraic consequences of this, which we plan to elaborate on in joint work with Nick Bezhanishvili [2].

The structure of the note is as follows: in Section 2 we recall some basic preliminaries, and fix some notation moving forward. Section 3 contains the main technical tools, in the form of the description of the free Heyting algebra generated by a distributive lattice preserving a given set of relative pseudocomplements; in Section 4 we showcase some basic applications of the aforementioned theory to the study of Heyting algebras: some well-known results, like the amalgamation theorem, are shown here through direct methods, as well as some seemingly new results, like the co-distributivity of the category of Heyting algebras. In Section 5 we look at the case of subvarieties of Heyting algebras and illustrate how these constructions can be fruitfully analysed there. In Section 6 we look at the corresponding relationship between the category of posets with p-morphisms and the category of posets with monotone maps. We conclude in Section 7 with some outlines of future work.

2 Preliminaries

2.1 Algebraic Background

We assume the reader is familiar with the basic theory of Boolean algebras and Distributive lattices (see for example [12]). Throughout we will use the term "distributive lattices" for *bounded* distributive lattices, i.e., our lattices will always have a bottom element (0) and a top element (1). We also assume the reader is familiar with Heyting algebras and their elementary theory (see e.g. [15]).

We recall the following constructions, which are part of the folklore of the subject (see [28] for an in-depth discussion):

Definition 1. Let X be an arbitrary set. Let $\mathbf{B}(X)$ be the Boolean algebra dual to the Stone space

$$F(X) := \mathbf{2}^X$$

where $\mathbf{2}$ is the two element Stone space, and $F(X)$ is given the product topology.

Similarly, let $\mathbf{D}(X)$ be the distributive lattice dual to the Priestley space

$$F_D(X) := \mathbf{2}^X$$

where $\mathbf{2}$ is this time the poset $0 < 1$, and $F_D(X)$ is given the product topology and the product ordering.

We then have the following well-known fact:

Proposition 2. Let X be an arbitrary set. Then:

1. $\mathbf{B}(X)$ is the free Boolean algebra on X many generators;
2. $\mathbf{D}(X)$ is the free distributive lattice on X many generators.

Given $\mathcal{D}$ a distributive lattice, and $A \subseteq D$, we write $\mathsf{Fil}(A)$ for the filter generatedd by A, and similarly, $\mathsf{Id}(A)$ for the ideal generated by A. We also recall that the following prime filter theorem holds for distributive lattices:

Theorem 3. *Let $\mathcal{D}$ be a distributive lattice, $F, I \subseteq D$ be respectively a filter and an ideal, such that $F \cap I = \varnothing$. Then there exists a prime filter $F' \supseteq F$ such that $F' \cap I = \varnothing$.*

2.2 Duality Theory

Throughout we assume the reader is familiar with Stone duality, as covered for example in [12]. Given a distributive lattice, we denote the representation map as

$$\varphi : \mathcal{D} \to \mathsf{ClopUp}(X_D)$$

where $\varphi(a) = \{x \in X_D : a \in x\}$.

We also note that for our purposes a given ordered topological space $(X, \leqslant)$ is called a *Priestley space* if:

- (X, τ) is a compact topological space.
- For each $x, y \in X$, if $x \nleqslant y$ then there is a clopen upset U such that $x \in U$ and $y \notin U$.

We say that a Priestley space X is an *Esakia space* if whenever $U \subseteq X$ is a clopen set, then ${\downarrow}U$ is likewise clopen.

A fact that will often use below is that if $(X, \leqslant)$ is a Priestley space, and $Y \subseteq X$ is a closed subset, then $(Y, \leqslant_{\restriction Y})$ is a Priestley space as well. We also note the following important, though easy to prove, fact:

Proposition 4. Let $\mathcal{D}$ be a distributive lattice, and let $a, b \in \mathcal{D}$ be arbitrary. Then if c is the relative pseudocomplement of a and b, then

$$\varphi(c) = X_D - {\downarrow}(\varphi(a) - \varphi(b)).$$

An important part of our work will deal with inverse limits of Priestley spaces. The following is clear from duality theory:

Proposition 5. Let $(\mathcal{D}_n, \iota_{i,j})$ be a chain of distributive lattices, connected by embeddings $i_{i,j} : \mathcal{D}_i \to \mathcal{D}_j$ for $i < j$ satisfying the compatibility laws. Let $(X_n, p_{i,j})$ be the inverse system of Priestley spaces one obtains by dualising all arrows. Then the directed union of the directed chain, $\mathcal{D}_\omega$, is dual to the projective limit X_ω of the inverse system. Moreover, if each X_n is an Esakia space, and each $p_{i,j}$ is a p-morphism, then X_ω is likewise an Esakia space.

We also recall the well-known construction of the *Vietoris hyperspace*, due to Leopold Vietoris [30]: let $(X, \leqslant)$ be a Priestley space. Let $V(X)$ be the set of all closed subsets of X. We give this set a topology by considering a subbasis consisting, for $U, V \subseteq X$ clopen sets:

$$[U] = \{C \in V(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in V(X) : C \cap V \neq \varnothing\}.$$

This is sometimes called the "hit-and-miss topology", and the resulting space is called the *Vietoris hyperspace*. On this space we define an order relation $C \preceq D$ if and ony if[1] $D \subseteq C$. Then we have the following:

Lemma 6. The space $(V(X), \preceq)$ is a Priestley space.

Proof. The fact that $V(X)$ is compact is a standard fact (see e.g. [14]), but we prove it here for completeness. assume that $V(X) = \bigcup_{i \in I}[U_i] \cup \bigcup_{j \in J}\langle V_j \rangle$ is a cover by clopen sets. Now consider $C = X - \bigcup_{i \in I} V_i$. If C is empty, then the sets V_i cover X and so finitely many of them cover X, say $V_{i_0}, ..., V_{i_n}$. Then if $A \in V(X)$, then A is non-empty and so contains some x which must lie in some V_{i_k} for $k \in \{1, ..., n\}$, and so $A \in \langle V_{i_k} \rangle$. So the sets $\{\langle V_{i_k} \rangle : k \in \{1, ..., n\}\}$ form a finite subcover.

Otherwise, we have that $C \neq \varnothing$, so $C \in V(X)$, and since $C \notin \langle V_j \rangle$ for each i, by definition $C \in [U_i]$ for some i. Since $X - \bigcup_{i \in I} V_i \subseteq U_i$ then $X - U_i \subseteq \bigcup_{i \in I} V_i$, so by compactness, $X - U_i \subseteq V_{i_0} \cup ... \cup V_{i_n}$. Hence consider the finite subcover $\{[U_i]\} \cup \{\langle V_{i_j} \rangle : j \in \{1, ..., n\}\}$. If D is any closed set, and $D \subseteq U_i$, then we are done; otherwise, $D \cap X - U_i \neq \varnothing$, so $D \cap V_{i_0} \cup ... \cup V_{i_n} \neq \varnothing$, from whence the result follows.

Now we prove the Priestley Separation axiom. Assume that $C \npreceq D$ are closed subsets. Since the space X is a Stone space, we know that $C = \bigcap_{i \in I} V_i$ and $D = \bigcap_{j \in J} W_j$ where V_i, W_j are clopen. Hence there must be some i such that $C \subseteq V_i$ and $D \nsubseteq V_i$; i.e., $C \in [V_i]$ and $D \notin [V_i]$. Since $[V_i]$ is clearly a clopen upset under this order, we verify the Priestley separation axiom. ∎

3 Free Constructions of Heyting Algebras over Distributive Lattices

In this section we explain how one can construct a Heyting algebra freely from any given distributive lattice. This construction generalises the case of Heyting algebras freely generated by finite distributive lattices, as originally analyzed by Ghilardi [16]. The key tools at hand will be the Priestley/Esakia duality of distributive lattices.

3.1 Conceptual Idea of the Construction

Before diving into the details, we will provide an informal explanation of the construction by Ghilardi which justifies the technical developments, following the core of the discussion from [7]. We urge the reader to consult this section for intuition whilst going through the details of the next section.

Suppose that D is a finite distributive lattice, and that we wish to construct a Heyting algebra from D. Then first we must (1) freely add to D implications of the form $a \to b$ for $a, b \in D$. This is the same as freely generating a distributive lattice out of D^2, and considering its coproduct with D. Dually, if X is the dual poset to D, this will then be the same as considering

$$X \times \mathcal{P}(X \times X),$$

[1] The reason this order is reverse inclusion has to with the choice to use *upsets* rather than downsets; in [16], the author uses downsets, which is why the swap no reversion is needed.

given that the free distributive lattice on a set of pairs of generators X will be dual to $\mathcal{P}(X \times X)$ (see Section 2). However, we need (2) to impose axioms forcing these implications to act like relative pseudocomplements. A first move is to impose the axiom of a Weak Heyting Algebra:

1. $a \to a = 1$;
2. $(a \vee b) \to c = a \to c \wedge b \to c$;
3. $a \to (b \wedge c) = a \to b \wedge a \to c$;
4. $a \to b \wedge b \to c \leqslant a \to c$.

Applying a quotient under these axioms, will dually yield (see [7, Theorem 3.5]),

$$\mathcal{P}(X),$$

Now, reformulating this slightly, one can see that the upsets of $\mathcal{P}(X)$ are of the form

$$[U] = \{C \subseteq X : C \subseteq U\}$$

for U a subset of X. This then provides an expansion of our lattice, since we can consider the map:

$$\begin{aligned} i_0 : D &\to \mathsf{Up}(\mathcal{P}(X)) \\ a &\mapsto [\varphi(a)], \end{aligned}$$

which is easily seen to be injective, and a meet-homomorphism preserving the bounds.

The fact that we want to obtain a genuine relative pseudocomplement, means we need to impose further axioms, which implies throwing out some of the elements from $\mathcal{P}(X)$. The additional fact that we would want the map i_0 to be a distributive lattice homomorphism suggests a way of doing this: make it so that for each pair of upsets U, V, $[U] \cup [V] = [U \cup V]$. After some thought one can then be lead to consider *rooted* subsets of X, obtaining the set $\mathcal{P}_r(X)$. Some verifications show that this will indeed be the poset that is needed: that

$$i_0 : D \to \mathsf{Up}(\mathcal{P}_r(X))$$

will be a distributive lattice embedding as desired, and that it will contain relative pseudocomplements for the elements from D, namely, for $a, b \in D$, the element $[\Box(-\varphi(a) \cup \varphi(b))]$.

Now at this point we will have added all implications to elements $a, b \in D$, obtaining a distributive lattice D_1, but all the new implications added might not in turn have implications between themselves. So we need to (3) iterate the construction, infinitely often, to add all necessary implications. However, the final complication is that each step of this construction adds implications to every element in the previous lattice. So in particular, D_2 will contain a fresh relative pseudocomplement, $[a] \to [b]$. If we let the construction run infinitely often in this way, it could be that in the end no element would be the relative pseudocomplement of a and b, so we need to ensure that on the second iteration, the previously added relative pseudocomplements are preserved, i.e.:

$$i_1(i_0(a) \to i_0(b)) = i_1(i_0(a)) \to i_1(i_0(b))$$

In other words, we need the map i_1 to preserve the relative pseudocomplements of the form $i_0(a) \to i_0(b)$. It is this need which justifies the notion of a g-open subset detailed below, and leads to us considering, at last, as our one step construction, the poset

$$\mathcal{P}_g(X)$$

as will be explained in the next section, where g is some order-preserving map which serves to index the relative pseudocomplements which are to be preserved.

With this in mind, we turn in the next section to the technical details involved in this generalization.

3.2 G-Open Subsets and Vietoris Functors

We begin by adapting the notion of g-openness to our setting. Let X, Y be two Priestley spaces. We recall that a Priestley morphism $f : X \to Y$ is an order-preserving and continuous map. By Priestley duality, such maps are dual to distributive lattice homomorphisms $f^{-1} : \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$.

Definition 7. Let X, Y, Z be Priestley spaces, and $f : X \to Y$ and $g : Y \to Z$ be Priestley morphisms. We say that f is *open relative to* g (g-open for short) if f^{-1} preserves relative pseudocomplements of the kind $g^{-1}[U] \to g^{-1}[V]$ where $U, V \in \mathsf{ClopUp}(Z)$.

The following is a condition equivalent to being g-open, expressed purely in terms of the order:

$$\forall a \in X, \forall b \in Y, (f(a) \leqslant b \implies \exists a' \in X, (a \leqslant a' \;\&\; g(f(a')) = g(b)). \tag{*}$$

Lemma 8. Given $f : X \to Y$ and $g : Y \to Z$, we have that f is g-open if and only if f satisfies condition (*).

Proof. Asssume that f satisfies (*). We want to show that $f^{-1}[g^{-1}[U] \to g^{-1}[V]] = f^{-1}g^{-1}[U] \to f^{-1}g^{-1}[V]$. Note that the left to right inclusion always holds because f is a Priestley morphism. So assume that $x \in f^{-1}g^{-1}[U] \to f^{-1}g^{-1}[V]$. Suppose that $f(x) \leqslant y$, and $y \in g^{-1}[U]$. By assumption, there is some x' such that $x \leqslant x'$ and $g(f(x')) = g(y)$; hence $f(x') \in g^{-1}[U]$, so by assumption, $x' \in f^{-1}g^{-1}[U]$, and so, $x' \in f^{-1}g^{-1}[V]$. This means that $f(x') \in g^{-1}[V]$, so $y \in g^{-1}[V]$. This shows what we want.

Now assume that $p = f^{-1}$ is g-open, and $q = g^{-1}$. Assume that $f(x) \leqslant y$ where $x \in X$ and $y \in Y$. By duality, and abusing notation, this means that $p^{-1}[x] \subseteq y$. Consider $\mathsf{Fil}(x, \{p(q(a)) : q(a) \in y\})$ and $\mathsf{Id}(\{p(q(b)) : q(b) \notin y\})$. We claim these do not intersect. Because otherwise, for some $c \in x$, $c \wedge p(q(a)) \leqslant p(q(b))$. Hence $c \leqslant p(q(a)) \to p(q(b))$, since these exist and are preserved by p, and so $c \leqslant p(q(a) \to q(b))$. So $p(q(a) \to q(b)) \in x$. Hence $q(a) \to q(b) \in p^{-1}[x]$, and so $q(a) \to q(b) \in y$, a contradiction. Hence by the Prime filter theorem (see Theorem 3), we can extend x to a filter x' which does not intersect the presented ideal. By definition, working up to natural isomorphism, we then have that $g(f(x')) = g(y)$, which was to show. ■

Now given $g : X \to Y$ a Priestley morphism, and $S \subseteq X$ a closed subset, we say that S is *g-open* (understood as a poset with the restricted partial order relation) if the inclusion is itself g-open. This means by the above lemma that S is g-open if the following condition holds:

$$\forall s \in S, \forall b \in X (s \leqslant b \implies \exists s' \in S(s \leqslant s' \;\&\; g(s') = g(b)).$$

Following Ghilardi, this can be thought of as follows: if we think of X as represented by fibers coming from g, then whenever $\uparrow s$ meets an element of any fiber, then $\uparrow s \cap S$ must actually contain an element of that fiber. With this intuition, it is not difficult to show that if x is arbitrary and S is g-open, then $S \cap \uparrow x$ is g-open as well.

At this stage we would like to point out to the reader that the fact that we pick a given unique g is totally incidental; in fact, whilst the results of this section will be proved for a single g, one could take any number of continuous morphisms and obtain the same results. We will not pursue such considerations further in this paper, however.

Throughout, fix $g : X \to Y$ a continuous and order-preserving map (the case for several such maps being preserved is entirely similar). Recall from Subsection 2.2 that $(V(X), \preceq)$ is a Priestley space. From this space we can move closer to the space we will be interested, by first considering $V_r(X) \subseteq V(X)$, the space of rooted closed subsets, with the induced order and the subspace topology. On this we can prove the following[2]:

Lemma 9. The space $(V_r(X), \preceq)$ is a Priestley space.

[2] This is stated without proof in [8, Lemma 6.1]. The key idea of the proof below was communicated to me by Mamuka Jibladze, to whom I am greatly indebted.

Proof. It suffices to show that $V_r(X)$ is closed. For that purpose, consider the following two subsets of $V(X) \times X$:

$$LB(X) = \{(C, r) : C \subseteq \uparrow r\} \text{ and } \ni_X = \{(C, r) : r \in C\}.$$

We note that $\ni_X$ is closed, since it is the pullback of the continuous map $\{\} : X \to V(X)$ along the second projection of $\leqslant \subseteq V(X) \times V(X)$ (which is a closed relation, since $V(X)$ is Priestley by Lemma 6). On the other hand, $LB(X)$ can also be seen to be closed, through a direct argument: suppose that $(C, r) \notin LB(X)$. By definition, then, $C \nsubseteq \uparrow r$, so there is a point $y \in C$ such that $r \nleqslant y$. By the Priestley separation axiom, there is a clopen upset U such that $r \in U$ and $y \notin U$. So consider

$$S = \langle X - U \rangle \times U.$$

This is clearly an open subset of $V(X) \times X$, and (C, r) belongs there by hypothesis. But also, if $(D, k) \in S$, then $D - U \neq \varnothing$, so there is some point $m \notin U$; but since U is an upset, then we must have $k \nleqslant m$, i.e., $(D, k) \notin LB(X)$. This shows that $LB(X)$ is closed.

Now note that

$$V_r(X) = \pi_{V(X)}[LB(X) \cap \ni_{V(X)}].$$

Since π is a closed map (given it is a continuous surjection between Stone spaces), we then have that $V_r(X)$ is a closed subspace of $V(X)$, as desired. ∎

Finally, we will now refine the rooted subsets to the ones we are truly interested in – the g-open ones. The following lemma encapsulates the key fact about this.

Lemma 10. Assume that $g : X \to Y$ is an order-preserving and continuous map, and that X has relative pseudocomplements of sets of the form $g^{-1}[U]$. Then the subspace $V_g(X) \subseteq V_r(X)$ is a closed subspace, and hence is a Priestley space as well with the induced order.

Proof. Assume that M is not g-open. This means that if x is the root of M, there is some y such that $x \leqslant y$, but for each $k \in M$, $g(k) \neq g(y)$. Hence $g(y) \notin g[M]$, and hence $\{g(y)\} \cap g[M] = \varnothing$. Since g is a continuous map between Stone spaces, $g[M]$ is closed; hence, since this is a Priestley space, there is a clopen set of the form $-U_i \cup V_i$ for U_i, V_i clopen upsets, such that $g[M] \subseteq -U_i \cup V_i$, and $g(y) \notin -U_i \cup V_i$. Then we have that $M \subseteq g^{-1}[-U_i] \cup g^{-1}[V_i]$ and $M \cap \downarrow(g^{-1}[U_i] - g^{-1}[V_i]) \neq \varnothing$. Hence consider

$$S = [g^{-1}[-U_i] \cup g^{-1}[V_i]] \cap \langle \downarrow(g^{-1}[U_i] - g^{-1}[V_i]) \rangle.$$

Because X has relative pseudocomplements of sets of the form $g^{-1}[U]$, by Proposition 4 we have that $\downarrow(g^{-1}[U_i] - g^{-1}[V_i])$ will be clopen in X. Hence the set S is an intersection of subbasic sets. Moreover, M belongs there. Now if N belongs there, then first $N \subseteq g^{-1}[-U_i \cup V_i]$; and also, by downwards closure, its root k will be below some point y' in $g^{-1}[U_i] - g^{-1}[V_i]$. But then for each $m \in N$, $g(m) \neq g(k)$, so N is not g-open.

This shows that there is an open neighbourhood of M entirely contained outside of $V_g(X)$, i.e., that the set of g-open subsets is a closed subspace. ∎

We note that the hypothesis of the previous lemma – that X will have relative pseudocomplements of the form $g^{-1}[U] \to g^{-1}[V]$ – will always be satisfied in our contexts: this is how we think of g a way to parametrise those pseudocomplements which we wish to preserve.

Lemma 11. The map $r_g : V_g(X) \to X$ which sends a rooted, closed g-open subset to its root, is a continuous, order-preserving and surjective g-open map.

Proof. Simply note that if U is a clopen upset, $r_g^{-1}[U] = \{M : M \subseteq U\} = [U]$, and $r_g^{-1}[X - U] = \{M : M \cap X - U \neq \varnothing\} = \langle X - U \rangle$. The order-preservation is down to the order being reverse inclusion, and the surjectivity follows because, for each $x \in X$, $\uparrow x$ will always be g-open.

Moreover, note that r_g will be g-open: if $r_g(C) \leqslant y$, then $m \in C$ is such that $m \leqslant y$; so because C is g-open, there is some m' such that $m \leqslant m'$ and $g(m') = g(y)$. But then $C' := \uparrow m' \cap C$ is such that $C \preceq C'$ and $g(r_g(C')) = g(y)$, as desired. ∎

3.3 Free Heyting Algebras from Distributive Lattices

We will now put the tools developed in the previous section to use.

Definition 12. Let $g : X \to Y$ be a Priestley morphism. The *g-Vietoris complex* over X $(V^g_\bullet(X), \leqslant_\bullet)$, is a sequence

$$(V_0(X), V_1(X), ..., V_n(X), ...)$$

connected by morphisms $r_i : V_{i+1}(X) \to V_i(X)$ such that:

1. $V_0(X) = X$;
2. $r_0 = g$
3. For $i \geqslant 0$, $V_{i+1}(X) := V_{r_i}(V_i(X))$;
4. $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ is the root map.

Note that given a g-Vietoris complex, one can form the projective limit of the sequence, which, since Priestley spaces are closed under such a construction, will again be a Priestley space. We denote this limit by $V^g_G(X)$ (the G standing for "Ghilardi").

The purpose of such a construction lies in the universal property which is carried out over any one step, which we proceed to outline:

Lemma 13. Let $g : X \to Y$ be a continuous, order-preserving map. Then the pair $\langle \mathsf{ClopUp}(V_g(X)), r_g^{-1} \rangle$ has the following universal property: suppose we are given any other pair

$$\langle D, \mu : \mathsf{ClopUp}(X) \to D \rangle$$

such that

1. D is a distributive lattice containing relative pseudocomplements of the kind $\mu(C_1) \to \mu(C_2)$ for $C_1, C_2 \in \mathsf{ClopUp}(X)$.
2. $\mu(g^{-1}[D_1] \to g^{-1}[D_2]) = \mu(g^{-1}[D_1]) \to \mu(g^{-1}[D_2])$ for all $D_1, D_2 \in \mathsf{ClopUp}(Y)$.

Then there exists a unique lattice homomorphism $\mu' : \mathsf{ClopUp}(V_g(X)) \to D$ such that the triangle in Figure 1 commutes, and such that $\mu'(r^{-1}[C_1] \to r^{-1}[C_2]) = \mu(C_1) \to \mu(C_2) = \mu'(r^{-1}[C_1]) \to \mu'(r^{-1}[C_2])$ for all $C_1, C_2 \in \mathsf{ClopUp}(X)$.

$$\begin{tikzcd} & \mathsf{ClopUp}(X) \arrow[dl, "r_g^{-1}"'] \arrow[dr, "\mu"] & \\ \mathsf{ClopUp}(V_g(X)) \arrow[rr, "\mu'"] & & D \end{tikzcd}$$

Figure 1: Commuting Triangle of Distributive Lattices

The proof of this lemma is given by first realising what is the appropriate dual statement.

Lemma 14. The property of Lemma 13 is equivalent to the following: given a Priestley space Z with a g-open continuous and order-preserving map $h : Z \to X$, there exists a unique r_g-open, continuous and order-preserving map h' such that the triangle in Figure 2 commutes.

Proof. This follows immediately from Lemma 8 by dualising the statements. ■

$$
\begin{tikzcd}
Z \arrow[rr, "h'"] \arrow[dr, "h"'] & & V_g(X) \arrow[dl, "r"] \\
& X &
\end{tikzcd}
$$

Figure 2: Commuting Triangle of Priestley spaces

Proof. (Of Lemma 13) We show the property from Lemma 14. Given Z, X Priestley spaces as given, the definition of h' is totally forced by the commutativity of the triangle. Indeed, given $a \in Z$, let

$$h'(a) = \{h(b) : a \leqslant b\}.$$

Note that $h'(a) = h[\uparrow a]$, and since h is continuous between Stone spaces, and hence a closed map, h' maps a to a closed subset; it is of course also rooted. And it is g-open as a subset, because h is g-open as a map. Clearly $r_g(h'(a)) = h(a)$. It is continuous, since $(h')^{-1}[[-U \cup V]] = \{a : h'(a) \in [-U \cup V]\}$, and this is the same as saying that $\uparrow a \subseteq h^{-1}[-U] \cup h^{-1}[V]$. Now since Z has relative pseudocomplements of these sets, then $\Box(-U \cup V)$ exists, and indeed

$$\{a : \uparrow a \subseteq h^{-1}[-U] \cup h^{-1}[V]\} = \{a : a \in \Box(h^{-1}[-U] \cup h^{-1}[V])\} = \Box(h^{-1}[-U] \cup h^{-1}[V]).$$

This shows continuity. It is also of course order-preserving, since if $a \leqslant b$, then $\uparrow b \subseteq \uparrow a$, so $h[\uparrow b] \subseteq h[\uparrow a]$, i.e., $h'(a) \leqslant h'(b)$. Finally it is also r_g-open: if $a \in Z$, and $h'(a) \leqslant M$ where $M \in V_g(X)$, then $M \subseteq h'(a)$, so if z is the root of M, then $z = h(c)$, where $a \leqslant c$. Then $r(h'(c)) = r(M)$, as desired. The uniqueness is enforced by the shape of the diagram. ∎

All of this allows us to prove the following key theorem:

Theorem 15. *Let $\mathcal{A}$ be a Heyting algebra, let $\mathcal{D}$ be a distributive lattice, and let $p : \mathcal{A} \to \mathcal{D}$ be a distributive lattice homomorphism which preserves the relative pseudocomplements from $\mathcal{A}$ (such that D contains them). Let $\mathcal{H}$ be the dual of $V_G^p(X)$. Then $\mathcal{H}$ is the Heyting algebra freely generated by D which preserves the relative pseudocomplements coming from p.*

Proof. By Proposition 5 we have that $\mathcal{H}$ is the directed union of the algebras $H_n(D)$, which are the duals of $V_n(X)$. First we note that $\mathcal{H}$ is a Heyting algebra when induced with the obvious relative pseudocomplement operation: given $a, b \in \mathcal{H}$, we can find them together at a given stage n, and hence at stage $n + 1$, they will have a relative pseudocomplement, which is henceforth preserved – hence the natural definition of the Heyting implication is well-defined.

The freeness amounts to proving that given any Heyting algebra $\mathcal{M}$ such that $k : \mathcal{D} \to \mathcal{M}$ is a map preserving the relative pseudocomplements coming from $\mathcal{A}$, there is a unique lifting of k to $\mathcal{H}$ to $\mathcal{M}$. By Lemma 13, we have that for each n there will be a unique map from $H_n(A)$ to $\mathcal{M}$, $q_n : H_n(D) \to \mathcal{M}$ which makes the colimit diagram commute, and which preserves the right relative pseudocomplements. So by the universal property, there will be a map

$$q : \mathcal{H} \to \mathcal{M},$$

which lifts all of these maps. Note that then q is not only a distributive lattice homomorphism, but indeed a Heyting homomomorphism: if $a, b \in \mathcal{H}$, we find $H_n(D)$ in which they both belong, and note that

$$q(a \to b) = q_n(a \to b) = q_n(a) \to q_n(b) = q(a) \to q(b),$$

taking all these equalities modulo the respective equivalence classes. Moreover, such a map is unique given the uniqueness of each of q_n and the uniqueness of the colimit map q. Hence this proves that $\mathcal{H}$ has the desired universal property. ∎

For the sequel, let us fix a piece of notation: given a Heyting algebra $\mathcal{A}$, a distributive lattice $\mathcal{D}$, and a map $p : \mathcal{A}_i \to \mathcal{D}$ preserving the relative pseudocomplements from $\mathcal{A}$, we let $\mathcal{H}(D, p)$ be the free Heyting algebra generated as above. When $\mathcal{A} = \mathbf{2}$ we often omit p, since the morphism is unique. We now proceed to prove that additionally, this construction is functorial.

Proposition 16. Let $f : \mathcal{D} \to \mathcal{D}'$ be a distributive lattice homomorphism, $\mathcal{A}$ a Heyting algebra, and let $g : \mathcal{A} \to \mathcal{D}$ and $g' : \mathcal{A} \to \mathcal{D}$ be two distributive lattice homomorphisms such that $g'f = g$ and f is g-open. Then there is a unique Heyting algebra homomorphism $\hat{f} : \mathcal{H}(D, g) \to \mathcal{H}(D', g')$ extending f, and moreover, this assignment is functorial.

Proof. Note that $f : D \to D'$ is a distributive lattice homomorphism, and by composing it with the inclusion $i : D' \to H(D')$ where $i' = (r_{g'})^{-1}$, we obtain a distributive lattice homomorphism if into a lattice possesing relative pseudocomplements of the kind $if(a) \to if(b)$ for $a, b \in D$ (since these will exist freely by construction of $H(D)$), and

$$i'f(g(a) \to g(b)) = i'(fg(a) \to fg(b)) = i'(g'(a) \to g'(b)) = i'g'(a) \to i'g'(b) = i'fg(a) \to i'fg(b),$$

where the first equality follows from f being g-open; the second from the commutativity of the diagram; the third from i' being g'-open, and the last again by commutativity of the diagram. Then by Lemma 13, we have a unique map $f^1 : \mathsf{ClopUp}(V_g(X)) \to \mathsf{ClopUp}(V_{g'}(X))$. such that $fi' = f^1 i$, where $i = (r_g)^{-1}$, which is additionally i-open. This process can naturally be iterated to create an infinite chain of distributive lattice homomorphisms as depicted in Figure 3.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "g"'] \arrow[dr, "g'"] & \\
\mathcal{D} \arrow[rr, "f"] \arrow[d, "i"'] & & \mathcal{D}' \arrow[d, "i'"] \\
H(\mathcal{D}) \arrow[rr, "f^1"] \arrow[d, "i^2"'] & & H(\mathcal{D}') \arrow[d, "(i')^2"] \\
H^2(\mathcal{D}) \arrow[rr, "f^2"] \arrow[d] & & H^2(\mathcal{D}') \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}$$

Figure 3: Infinite chain of liftings

Due to the properties of the colimits, this means that there is a map $f^\infty : \mathcal{H}(D, g) \to \mathcal{H}(D', g')$, which agrees with f^n on each restriction. This is certainly a distributive lattice homomorphism, but we claim that it is also a Heyting algebra homomorphism. To see this, note that if $a, b \in \mathcal{H}(D, g)$, then $a, b \in H^n(D)$ for some n; then

$$f^\infty(a \to b) = f^{n+1}(a \to b) = f^{n+1}(a) \to f^{n+1}(b) = f^\infty(a) \to f^\infty(b),$$

which shows the desired result. ■

As the reader might have guessed, the previous construction is always the left adjoint to a suitable inclusion of categories. We will now formally elaborate on this. Namely, given a Heyting algebra $\mathcal{A}$, consider the category $\mathcal{A}/\mathbf{DLat}$, which objects are pairs $(\mathcal{D}, p)$ of a distributive lattice and a relative pseudocomplement preserving homomorphism $p : \mathcal{A} \to \mathcal{D}$, and where the morphisms are distributive lattice homomorphisms making the obvious triangle commute; this is a supercategory of the category $\mathcal{A}/\mathbf{Heyt}$, where all objects and morphisms are Heyting. Denote by $G : \mathcal{A}/\mathbf{Heyt} \to \mathcal{A}/\mathbf{DLat}$ the inclusion functor. Then we have the following:

Theorem 17. *Given $\mathcal{A}$ a Heyting algebra, the functor $\mathcal{H}^{\mathcal{A}}(\mathcal{A}) : \mathcal{A}/\mathbf{DLat} \to \mathcal{A}/\mathbf{Heyt}$ is the left adjoint to G.*

By duality, we obtain, in the special case where $\mathcal{A} = \mathbf{2}$, the following:

Theorem 18. *The functor $V_G : \mathbf{Pri} \to \mathbf{Esa}$ is right adjoint to the inclusion $I : \mathbf{Esa} \to \mathbf{Pri}$. By duality, the functor $V_G^* : \mathbf{DL} \to \mathbf{HA}$ is left adjoint to the forgetful functor $I : \mathbf{HA} \to \mathbf{DL}$.*

4 Some Applications to the theory of Heyting Algebras

4.1 Free Heyting Algebras on an arbitrary number of Generators and Tree Representations

As a first application of Theorem 15, we show how to obtain a description of the free Heyting algebra on any number of generators. Let X be an arbitrary set, and recall Definition 1.

Theorem 19. *The algebra $\mathcal{H}(\mathbf{D}(X))$ is the free Heyting algebra on X many generators.*

Proof. It suffices to verify the universal property. Suppose that $k : X \to \mathcal{M}$ is a map to a Heyting algebra $\mathcal{M}$. By the universal property of free distributive lattices, there is then a unique lift to a distributive lattice homomorphism $\tilde{k} : \mathbf{D}(X) \to \mathcal{M}$. In turn, by Theorem 15, there is a unique homomorphism $k'' : \mathcal{H}(\mathbf{D}(X)) \to \mathcal{M}$ which preserves the relative pseudocomplements coming from $\mathbf{2}$; but since every distributive lattice homomorphism preserves such relative pseudocomplements (since they amount to preserving the bounds), this shows that $\mathcal{H}(\mathbf{D}(X))$ is indeed the free Heyting algebra on X generators. ■

As is discussed [16], in specific cases, it might be useful to have a representation of our construction that is slightly more concrete. We will exemplify this with the case at hand: consider $V_G(2^X)$, the dual of $\mathcal{H}(\mathbf{D}(X))$. Then its elements are of the form:

$$(C_0, C_1, ...)$$

where C_0 is an element from 2^X, and hence, a subset of X; C_1 is a closed collection of such subsets; and so on. We can visualise such elements in the form of trees: to each C_{n+1} we associate a labelling function, taking values in 2^X, and which is defined as:

1. $\psi(C_0) = \langle C_0, \varnothing \rangle$;
2. $\psi(C_{n+1}) = \langle r_0 \cdot ... \cdot r_n(C_{n+1}), \{\psi(B_n)\}_{B_n \in C_{n+1}} \rangle$.

As such, the elements we see in the projective limit appear as the "leftmost branch" of such a tree, which we can think of as the limiting result of building such a tree iteratively. In this case, such elements provide tree-models over the set X, with each label consisting of some subset of the propositions which happens to *not* be true at that world (which again follows from the reverse inclusion).

4.2 Coproducts of Heyting Algebras

As a further application, we can provide an explicit description of the coproduct of two Heyting algebras. This follows generally from the fact that left adjoints preserve all colimits; nevertheless we give an explicit proof which highlights the key properties used.

Theorem 20. *Let $\mathcal{A}, \mathcal{B}$ be two Heyting algebras. Let $\mathcal{A} \oplus \mathcal{B}$ be the tensor sum of $\mathcal{A}, \mathcal{B}$ (i.e., the dual of $X_A \times X_B$, the product of the Priestley spaces). Then $\mathcal{H}(\mathcal{A} \oplus \mathcal{B})$ is the coproduct of $\mathcal{A}, \mathcal{B}$.*

Proof. We verify the universal property. Assume that $k_1 : A \to H$ and $k_2 : B \to H$ are Heyting algebra homomorphisms. Dually, this means that $r_1 : X_H \to X_A$ and $r_2 : X_H \to X_B$ are p-morphisms. Because $X_A \times X_B$ is the Priestley product, we get a universal, unique map $v : X_H \to X_A \times X_B$. Note that v is p-open for p the terminal map. Hence by Theorem 15, there is a unique Heyting algebra homomorphism $k : \mathcal{H}(A \oplus B) \to \mathcal{H}$. Because of the uniqueness of all constructions involved, it is then easy to see that any other homomorphism in these conditions would need to coincide with v', showing that $\mathcal{H}(A \oplus B)$ is the coproduct, as desired. ∎

The reader will observe that, in line with our comments, the former is not limited to binary products, which means that we can provide an explicit description of arbitrary products of Esakia spaces as well. More interestingly, having such an explicit description of the coproduct allows us to prove some purely categorical properties of the category of Heyting algebras:

Theorem 21. *The category of Heyting algebras is co-distributive, i.e.,* $A \coprod (B \times C) \cong (A \coprod B) \times (A \coprod C)$*.*

Proof. Note that dually, $(\mathcal{A} \coprod \mathcal{B}) \times (\mathcal{A} \coprod \mathcal{C})$ will be the disjoint union of $X_{A \coprod B}$ and $X_{A \coprod C}$. We will then show that the dual of $\mathcal{A} \coprod (\mathcal{B} \times \mathcal{C})$ is isomorphic to this space as well.

Indeed, consider A and $B \times C$. Note that $X_A \times (X_B + X_C)$, understood as a Priestley space, is isomorphic to $(X_A \times X_B) + (X_A \times X_C)$. Moreover, note that essentially because the union is disjoint, letting p be the map to the one element poset:

$$V_p((X_A \times X_B) + (X_A \times X_C)) \cong V_p(X_A \times X_B) + V_p(X_A \times X_C).$$

Indeed, since the subsets are rooted, they must lie entirely in one disjoint factor or the other. Moreover, note that the root map likewise factors appropriately.

Hence the projective limit $V_G(X_A \times (X_B + X_C))$ will be isomorphic to

$$\varprojlim_{n \in \omega} V_n(X_A \times X_B) + V_n(X_A \times X_C).$$

Hence we show that

$$\varprojlim_{n \in \omega} V_n(X_A \times X_B) + V_n(X_A \times X_C) \cong (\varprojlim_{n \in \omega} V_n(X_A \times X_B)) + (\varprojlim_{n \in \omega} V_n(X_A \times X_C).$$

To see this, note that if x belongs to the first set, and $x(n) \in V_n(X_A \times X_B)$, then since $x(n+1)$ must have its root in this set, $x(n+1) \in V_{n+1}(X_A \times X_B)$; this means that $x \in \varprojlim_{n \in \omega} V_n(X_A \times X_B)$. Conversely, if x belongs to one of the two sets, then certainly it belongs to the projective limit of the disjoint unions. It is thus not difficult to see that this is in an isomorphism. But this means that the dual of $\mathcal{A} \coprod (\mathcal{B} \times \mathcal{C})$ is isomorphic to the disjoint union of the dual of $\mathcal{A} \coprod \mathcal{B}$ and the dual of $\mathcal{A} \coprod \mathcal{C}$, which is exactly what we wanted to prove. ∎

4.3 Pushouts and Co-Amalgamation

Making use of the explicit description of the product given before, we can likewise provide, dually, a description of the pullback of two Esakia spaces.

Theorem 22. *Let X, Y, Z be Esakia spaces, and let $f : X \to Z$ and $g : Y \to Z$ be two p-morphisms. Let $X \times_Z Y$ be the pullback in the category of Priestley spaces. Then the pullback of this diagram consists of*

$$X \otimes_Z Y := V_G(X \times_Z Y)$$

Using this pullback we can obtain an explicit proof of the amalgamation property for Heyting algebras, which is "constructive" in the sense that it is brought about by categorical, rather than model-theoretic or logical, considerations.

Theorem 23. *The variety of Heyting algebras has the amalgamation property.*

Proof. Dually, this amounts to showing that whenever we have a cospan diagram $f : X \to Z$ and $g : Y \to Z$, where both f and g are surjective, then there is some W which is a cone for this diagram, and where the maps to X and Y are again surjective. Note that forming the pullback in the category of Priestley spaces, the projection maps

$$X \xleftarrow{\pi_X} X \times_Z Y \xrightarrow{\pi_Y} Y$$

are surjective, since for each $x \in X$, $f(x) = g(y)$ for some $y \in Y$, by surjectivity of g, and vice-versa. Certainly then the unique lifting from $V_G(X \times_Z Y)$ to the projections will likewise be surjective, and both will be p-morphisms, showing that amalgamation holds. ∎

5 Subvarieties of Heyting algebras

The construction just outlined can also be adapted to handle other varieties of Heyting algebras. In this section we illustrate this by providing analogous free constructions of KC-algebras and LC-algebras; we leave a detailed analysis of the full scope of the method for future work.

KC-algebras are Heyting algebras axiomatised with an additional axiom

$$\neg p \vee \neg\neg p;$$

they are also sometimes called WLEM-algebras (For Weak Excluded Middle), DeMorgan Algebras, amongst several other names. They appear importantly in settings like Topos Theory (see e.g. [10]). In turn, LC-algebras are Heyting algebras axiomatised by

$$p \to q \vee q \to p;$$

they are sometimes called "Gödel algebras" or "Gödel-Dummett algebras" as well as "prelinear Heyting algebras" (the latter is due to the fact that the subdirectly irreducible elements in such a variety are chains).

5.1 KC-algebras

In order to generalise our construction to KC, we will need first the notion of a KC-distributive lattice. A dual perspective can illustrate what these should be: they should be the Priestley spaces which sit on a directed poset, precisely the distributive lattices over which the structure of a KC-algebra can be built; as it happens, this admits a simple dual description:

Definition 24. Let $(X, \leqslant)$ be a Priestley space. Let $R = (\leqslant \cup \leqslant^{\frown})^*$.We say that a subset C is *order-connected* if for each $x, y \in C$, we have xRy. We say that C is an *order-component* if it is a maximal order-connected subset.

Proposition 25. Let X be a Priestley space. Then the following are equivalent:

1. X is directed as a poset;
2. If $C \subseteq X$ is an order-component, then C has a unique maximal element.

Proof. It is clear, since each poset can be decomposed into its order-components, that if each order-component has a unique maximal element, then it is directed. Now assume that $(X, \leqslant)$ is directed. First, note that order-components are closed subsets: if $x \notin C$, then pick a clopen upset containing x, and this will provide an open neighbourhood separate from C. So by the theory of Priestley spaces, $\max(C) \neq \varnothing$. Now if $x, y \in \max(C)$, and they are different, since C is an order component, then there must be some point z such that $z \leqslant x$ and $z \leqslant y$ – which contradicts directedness. Hence we have the result. ∎

We will use the above lemma freely, and refer to Priestley spaces in these conditions simply as "directed".

Let us first outline our construction in algebraic terms. Given an appropriate distributive lattice $\mathcal{D}$, consider the complex:

$$(\mathcal{D}, H(\mathcal{D}), H^2(\mathcal{D}), ..., H^n(\mathcal{D})).$$

where we denote, as usual, the inclusions as $i_{i,j}$. In order to generate a KC algebra, we need to quotient this complex at appropriate steps. Namely, we need to replace $H^2(\mathcal{D})$ by the following:

$$K_C(\mathcal{D}) = H^2(\mathcal{D})/\{(\neg\neg i_{0,2}(a) \vee \neg i_{0,2}(a), \top) : a \in \mathcal{D}\}.$$

We then construct the alternative complex:

$$(\mathcal{D}, H(\mathcal{D}), K_C(H(\mathcal{D})), K_C^2(H(\mathcal{D})), ...).$$

Note that the limit of this construction will certainly be a Heyting algebra, and indeed, it will be a KC-algebra: for any choice a of value for p, there is a step where $\neg\neg a \vee \neg a$ is made equivalent to $\top$. Hence the question lies whether such a construction is still universal.

For that purpose we will find a dual meaning to forming the above quotient.

Definition 26. Let $g : X \to Y$ be a continuous and order-preserving map. Let $C \in V_{r_g}(V_g(X))$ be an element of the second stage of the g-Vietoris complex over X. We say that C is *well-directed* if whenever $D, D' \in C$, we have that $\uparrow D \cap \downarrow D' \neq \varnothing$. We write $V_{g,d}(V_g(X))$ for the subset of $V_{r_g}(V_g(X))$ consisting of the well-directed elements.

The motivation for this definition is the following:

Proposition 27. Let $g : X \to Y$ be a continuous and order-preserving function, and let $\mathcal{D} = X^*$. Then:

$$K_C(H(\mathcal{D}))^* \cong V_{g,K}(V_g(X))$$

Proof. By duality, we know that $K_C(H(\mathcal{D}))^*$ is obtained by considering all elements of $V_{r_g}(V_g(X))$ which satisfy the added axioms. Then we will show that for each such C, $C \Vdash \neg\neg[U] \vee \neg[U]$ if and only if C is well-directed.

To see this, note that, formally, if $C \nVdash \neg\neg[U] \vee \neg[U]$ for some clopen upset $U \subseteq X$; this means that $C \notin [-[-U]] \cup [[-U]]$. In other words, there is some $D \in C$ such that $D \notin -[-U]$, and some $D' \notin [-U]$. The former means that $D \subseteq -U$, and the later means that $D' \cap U \neq \varnothing$. Let $x \in D' \cap U$. Then $D'' = \uparrow x \cap D$ is such that $D' \leqslant D''$, and so because C is r_g-open, there is some $E \in C$ such that $D' \leqslant E$ and $r_g(D'') = r_g(E)$, i.e., x is the root of E. Since U is a clopen upset, $E \subseteq U$, and $D \subseteq -U$, so certainly

$$E \cap U = \varnothing.$$

This shows that C is not well-directed.

Conversely, assume that C is not well-directed. Let $D, D' \in C$ be such that $\uparrow D \cap \downarrow D' = \varnothing$. Using the properties of Priestley spaces, namely Strong Zero-Dimensionality, we can then find a clopen upset U such that $D \subseteq U$ and $D' \subseteq -U$; but this means precisely that $C \nVdash \neg\neg[U] \vee \neg[U]$ by the arguments above. ∎

Proposition 27 shows that the construction $V_{g,K}$ indeed yields a Priestley space, and so, that we can proceed inductively. In order to conclude our results, we will need to show, however, that this construction is free amongst the directed Priestley spaces. This amounts to showing the following:

Lemma 28. Let Z be a directed Priestley space, and let $h : Z \to X$ be a g-open, continuous and order-preserving map. Let $h^* : Z \to V_g(X)$ be the canonical map yielded by Lemma 13. Then the unique r_r-open, continuous and order-preserving map h' making the obvious triangle commute factors through $V_{g,K}(V_g(X))$.

Proof. Similar to Lemma 13 we have that the definition of h^{**} is entirely forced. All that we need to do is verify that given $x \in Z$, h^{**} is well-directed.

For that purpose, note that

$$h^{**}(x) = h^*[\uparrow x] = \{h^*(y) : x \leqslant y\}$$

Now assume that $x \leqslant y$ and $x \leqslant y'$. Because Z is directed, there is some z such that $y \leqslant z$ and $y' \leqslant z$. Then note that

$$h^*(z) \subseteq h^*(y) \cap h^*(y'),$$

since if $z \leqslant m$, then $y \leqslant m$ and $y' \leqslant m$, so $h(m) \in h^*(y) \cap h^*(y')$. This shows that $h^{**}(x)$ is well-directed, as desired. ■

Denote by $V_{GK}(X)$ the projective limit of the complex as before. Then using these constructions we obtain:

Theorem 29. *Let $\mathcal{A}$ be a Heyting algebra, let $\mathcal{D}$ be a distributive lattice, and let $p : \mathcal{A} \to \mathcal{D}$ be a distributive lattice homomorphism which preserves the relative pseudocomplements from $\mathcal{A}$ (such that D contains them). Let $\mathcal{K}$ be the dual of $V_{GK}(X)$. Then $\mathcal{K}$ is the* KC*-algebra freely generated by D which preserves the relative pseudocomplements coming from p.*

Proof. By our earlier remarks, we have that $\mathcal{K}$ is the directed union of Heyting algebras, and must satisfy the Weak Excluded Middle. To show freeness, suppose that $\mathcal{M}$ is a KC algebra and $k : \mathcal{D} \to \mathcal{M}$ is a map preserving the relative pseudocomplements coming from $\mathcal{A}$. Then using Lemma 28 systematically, we see that $\mathcal{M}$ will factor through $K^n_C(H(\mathcal{D}))$ for every n, and hence, a fortriori, through the limit. This shows the result. ■

Note that the above construction is not guaranteed to provide an *extension* of the algebra $\mathcal{D}$ – it might be that some elements are identified by the first quotient. However, it is not difficult to show that if $\mathcal{D}$ is a KC-distributive lattice, then it embeds into $\mathcal{K}(\mathcal{D})$ (by showing, for instance, that the composition with the root maps to $\mathcal{D}$ is always surjective). We also note that one could at this point show several of the same facts we did for Heyting algebras for KC-algebras.

5.2 LC-Algebras

We now consider the case of LC algebras. As is well-known, such a variety is locally finite. As such it will be interesting to both study the general construction and to outline how this might look when the starting algebras are finite.

First, let us say that a Priestley space X is *prelinear* if for each $x \in X$, $\uparrow x$ is a linear order.

Definition 30. Let $g : X \to Y$ be a Priestley morphism. We say that $C \in V_g(X)$ is a *linearised* closed, rooted and g-open subset, if C is a chain. Denote by $V_{g,L}(X)$ the set of linearised closed, rooted and g-open subsets.

As before, we define

$$L_C(\mathcal{D}) = H(\mathcal{D})/\{(i_{0,1}(a) \to i_{0,1}(b) \vee i_{0,1}(b) \to i_{0,1}(a), \top) : a, b \in \mathcal{D}\}$$

And we cosntruct the following complex:

$$(\mathcal{D}, L_C(\mathcal{D}), L^2_C(\mathcal{D}), ...)$$

Proposition 31. Let $g : X \to Y$ be a continuous and order-preserving function, and let $\mathcal{D} = X^*$. Then:

$$L_C(\mathcal{D})^* \cong V_{g,L}(X)$$

Proof. Assume that U, V are clopen subsets of X, and $C \nVdash [U] \to [V] \vee [V] \to [U]$. This amounts to saying that

$$C \nsubseteq -U \cup V \text{ and } C \nsubseteq -V \cup U.$$

In other words, $x \in C$ and $x \in U - V$ and $y \in C$ and $y \in V - U$. Since these sets are clopen subsets, this means that $x \nleqslant y$ and $y \nleqslant x$.

Conversely, assume that C is not linearised. Since $x \nleqslant y$, there is some clopen upset U such that $x \in U$ and $y \notin U$, and since $y \nleqslant x$, there is some clopen upset V such that $y \in V$ and $x \notin V$. These two subsets then witness that $C \nVdash [U] \to [V] \vee [V] \to [U]$. ∎

We now prove, again, an analogue to Lemma 13:

Lemma 32. Let Z be a prelinear Priestley space, and let $h : Z \to X$ be a g-open, continuous and order-preserving map. Then the unique r_g-open, continuous and order-preserving map h' making the obvious triangle commute factors through $V_{g,L}(X)$.

Proof. Simply note that since Z is prelinear, we have that $h[\uparrow x]$ is linear, given that $\uparrow x$ is as well. ∎

Denote by $V^{\infty}_{g,L}(X)$ the projective limit under the above complex. Then in the same way as before, we can prove:

Theorem 33. *Let $\mathcal{A}$ be a Heyting algebra, let $\mathcal{D}$ be a distributive lattice, and let $p : \mathcal{A} \to \mathcal{D}$ be a distributive lattice homomorphism which preserves the relative pseudocomplements from $\mathcal{A}$ (such that D contains them). Let $\mathcal{K}$ be the dual of $V^{\infty}_{g,L}(X)$. Then $\mathcal{L}$ is the LC-algebra freely generated by D which preserves the relative pseudocomplements coming from p.*

The following is within the scope of the methods of Ghilardi, using our modifications at hand for the case of LC. We found it in the context of analysing the infinite case, though a simpler finite combinatorial proof is presented here.

Consider a finite poset P. Because of local finiteness, we know that $V^{\infty}(P) \cong V^{n}(P)$ must hold for some finite n (otherwise the construction would add infinitely many new points). It is thus natural to wonder whether this construction can stabilise in a single step, i.e., whether after a single application of $V_L(X)$ the result will be the final result. Even for very simple posets it becomes apparent that this is not the case. However, surprisingly, two iterations seem to be enough.

Lemma 34. Let $C, D \in V^2_L(X)$, and assume that $V^2_L(X)$ is prelinear. Suppose that $r(C) = r(D)$ and that C and D are comparable. Then $C = D$.

Proof. Assume that $C \nleqslant D$, without loss of generality. By assumption, then $D < C$. Let $M \in D$ such that $M \notin C$. Hence we may assume that $M \neq r(C) = r(D)$. Consider $T = \uparrow M \cap D$. Then $D \leqslant T$. Then there are two cases:

1. If $T \leqslant C$, then $r(C) \in T$, a contradiction to the fact that $r(C) < M$.
2. If $C \leqslant T$, then $M \in C$, a contradiction to the assumption.

In both cases we find a contradiction, and hence conclude that $C = D$. ∎

Proposition 35. Let X be a finite poset. Assume that $V^2_L(X)$ is prelinear. Then $V^3_L(X) \cong V^2_L(X)$.

Proof. We prove that the only rooted linearised and r-open subsets of $V^2_L(X)$ are of the form $\uparrow x$ for $x \in V^2_L(X)$. Let $C \subseteq \uparrow x$ be a linearised rooted and r-open subset, such that $r(C) = x$. If $C \neq \uparrow x$ there is some y such that $x \leqslant y$ and $y \notin C$. Because C is r-open, there is some $y' \in C$ such that $r(y) = r(y')$. Now note that y' and y are comparable (they are both successors of x), so by Lemma 34, we have that $y = y'$, a clear contradiction. So we must have no such y must exist, and hence, that no C in these conditions can exist. We then conclude that every rooted linearised subset is of the form $\uparrow x$ for some x, which yields an easy isomorphism. ∎

Proposition 36. Let X be a finite poset. Then $V_L^2(X)$ is prelinear.

Proof. We show, by induction on the cardinality of $x \in V_L^2(X)$, that $\uparrow x$ is linear. Now first, notice that if x has cardinality 1, then it must be of the form $\{T_i\}$ for some $T_i \in V_L(X)$. Then note that T_i must be a singleton: if it had more than one element, say $T_i = \{x_0, ..., x_n\}$, then $T_i \leqslant \{x_i, ..., x_n\}$, for some $x_i \neq r(T_i)$, and this would violate r-openness. This means that the elements of cardinality 1 are exactly the elements of depth 1. Now assume that x is some element which is of depth 2, i.e., all of its proper successors are of depth 1. Then we claim it can have at most one such successor. Otherwise $\{a\} \in x$ and $\{b\} \in x$ for some elements a, b. But this is a contradiction, since x is assumed to be linearly ordered, and neither $\{a\} \leqslant \{b\}$ nor $\{b\} \leqslant \{a\}$. Hence, since x is of depth 2, it must contain some unique additional rooted set T_i, such that whenever $T_i \leqslant M$, then $r(M) = r(T_i)$ or $r(M) = \{c\}$ for the unique successor of x. This easily can be seen to entail that T_i must then have exactly two elements. Proceeding in the same way, if x is some element of depth 3, all of its proper successors are of depth 2, then it can have at most one such successor by similar arguments, and it must have exactly 3 elements: one of each cardinality from 1 to 3. The argument generalizes straightforwardly for any n. But this implies that then $\uparrow x$ is linear for any x. ∎

Putting these Propositions together, we obtain the following result, which is interesting from several points of view:

Theorem 37. *Let X be a finite poset. Then $V_L^2(X) \cong V_L^\infty(X)$.*

Proof. By Proposition 36 we know that $V_L^2(X)$ will be prelinear; and by Proposition 35 we have that iterating the construction past the second stage does not generate anything new. Hence for each $n > 2$, $V_L^n(X) \cong V_L^2(X)$, meaning that the projective limit will be likewise isomorphic. ∎

The above theorem, coupled with some of the theory we have developed for Heyting algebras, and which could easily be generalised for Gödel algebras, can have some computational uses: given two Gödel algebras, to compute their coproduct, we can look at their dual posets, form the product, and apply V_L^2, to obtain the resulting product. This provides an alternative perspective from the recursive construction discussed in [11, 1]. Moreover, it provides the following interesting property of Gödel logic:

Theorem 38. *Let $\varphi(p_0, ..., p_n)$ be a formula in the language of* IPC*. Then over* LC *φ is equivalent to a formula ψ, over the same propositional variables, of implication degree* 2*.*

6 Categories of Posets with P-morphisms

We now turn our attention to how the questions from the previous sections can be understood in a "discrete" setting – i.e., by considering posets, rather than Esakia spaces. One key motivation for this comes from the logical setting, where such discrete duals correspond to the "Kripke semantics" of modal and superintuitionistic logics. As we will see, a construction analogous to the V_G functor can be provided, relating the category of *Image-Finite Posets* with p-morphisms, and the category of posets with monotone maps. To justify it, we begin by outlining the relationship between such a category and the category of finite posets with p-morphisms, and the general phenomenon of profiniteness. We then describe the full construction.

6.1 Pro- and Ind-Completions of categories of finite objects

To explain the questions at hand, we recall the picture, outlined in [19, Chapter VI], relating categories of finite algebras and several of their completions. To start with, we have **FinBA**, the category of finite Boolean algebras, and its dual, **FinSet**. For such categories, there are two natural "completions":

- The *Ind-completion*, $\mathsf{Ind}(-)$, which freely adds all directed colimits;
- The *Pro-completion*, $\mathsf{Pro}(-)$, which freely adds all codirected limits.

For categories of algebras, the Pro-completion can equivalently be seen as consisting of the profinite algebras of the given type, whilst the Ind-completion can be seen as obtaining all algebras of that same type. As such, we have that:

$$\mathsf{Ind}(\mathbf{FinBA}) = \mathbf{BA} \text{ and } \mathsf{Pro}(\mathbf{FinBA}) = \mathbf{CABA}$$

where **BA** is the category of all Boolean algebras with Boolean algebra homomorphisms, and **CABA** is the category of complete and atomic Boolean algebras with complete homomorphisms. Correspondingly, we have that

$$\mathsf{Ind}(\mathbf{FinSet}) = \mathbf{Set} \text{ and } \mathsf{Pro}(\mathbf{FinSet}) = \mathbf{Stone}.$$

The facts that $\mathbf{Set}^{op} \cong \mathbf{CABA}$ (Tarski duality) and $\mathbf{Stone}^{op} \cong \mathbf{BA}$ (Stone duality) then amount to the basic relationship between Ind and Pro-completions, namely, for any category $\mathbf{C}$:

$$(\mathsf{Ind}(\mathbf{C}^{op}))^{op} \cong \mathbf{Pro}(\mathbf{C}) \text{ and } (\mathsf{Pro}(\mathbf{C}^{op}))^{op} \cong \mathsf{Ind}(\mathbf{C}).$$

A similar picture can be drawn for distributive lattices: there the relevant category of finite objects is the category **FinPos** of finite posets with monotone maps, which is dual to **FinDL**, finite distributive lattices with lattice homomorphisms, and we have that

$$\mathsf{Ind}(\mathbf{FinDL}) = \mathbf{DL} \text{ and } \mathsf{Pro}(\mathbf{FinDL}) = \mathbf{CCJDL}$$
$$\mathsf{Ind}(\mathbf{FinPos}) = \mathbf{Pos} \text{ and } \mathsf{Pro}(\mathbf{FinPos}) = \mathbf{Pries},$$

where **DL** is the category of all distributive lattices with lattice homomorphisms; **CCJDL** is the category of complete and completely join-prime generated distributive lattices with complete lattice homomorphisms; **Pos** is the category of all posets with monotone maps; and **Pries** is the category of Priestley spaces with Priestley morphisms.

If we are interested in Heyting algebras, it is not hard to see that the category of finite Heyting algebras with Heyting algebra homomorphisms is dual to the category $\mathbf{FinPos}_p$ of finite posets with p-morphisms. The Ind-completion of **FinHA** is of course still **HA**, and what we have shown above is that the inclusion of **HA** into **DL** admits a left adjoint. So if we wish to use this abstract perspective to "guess" a discrete category, the natural place to look is the Ind-completion of $\mathbf{FinPos}_p$. Such objects are those which are directed colimits of finite posets via p-morphisms – but this means that the finite posets must embed into the top of the object, i.e., such objects must be image-finite posets, in other words

$$\mathsf{Ind}(\mathbf{FinPos}_p) \cong \mathbf{ImFinPos}$$

where **ImFinPos** is the category of image-finite posets with p-morphisms. And indeed, we have that such a category is, by the main results from [4], dual to the category of profinite Heyting algebras, as desired. Indeed, a similar picture to this one has been discussed for finite modal algebras, and their duals, finite Kripke frames, in the recent preprint [13], where the above dualities are briefly sketched [3].

As a result of this relationship, it is natural to look for an adjunction holding between the inclusions of **ProHA** into **CCJDL**, or equivalently, for a right adjoint to the inclusion of **ImFinPos** into **Pos**. This is what we outline in the next section.

6.2 An Adjunction between the category of Image-Finite posets and Posets with Monotone Maps

Our constructions in this section mirror very much the key constructions from Section 3. Indeed, given a poset $g : P \to Q$, denote by

$$P_g(X) = \{C \subseteq X : C \text{ is rooted, } g\text{-open, and finite }\}.$$

[3]Implicit in the discussion above is the Raney duality between complete and completely join-prime generated distributive lattices, and its restriction to Heyting algebras. Whilst this result is very well-known and often mentioned in the literature, and traces its origins to the work of Raney, De Jongh and Troelstra, a concise proof of it does not seem to be available in the literature; in the Appendix we provide such a proof, for completeness

As usual when g is the terminal map to the one point poset, we write P_r. Also note that regardless of the structure of X, $P_g(X)$ will always be an image-finite poset, though in general the root maps will not admit any sections.

Definition 39. Let P, Q be posets with $g : P \to Q$ a monotone map. The *g-Powerset complex* over P, $(P^g_\bullet(X), \leqslant_\bullet)$ is a sequence

$$(P_0(X), P_1(X), ...)$$

connected by monotone maps $r_i : P_{i+1}(X) \to P_i(X)$ such that:

1. $P_0(X) = X$;
2. $r_0 = g$;
3. For $i \geqslant 0$, $P_{i+1}(X) \coloneqq P_{r_i}(P_i(X))$;
4. $r_{i+1} \coloneqq r_{r_i} : P_{i+1}(X) \to P_i(X)$ is the root map.

We write $P^g_G(P^g_\bullet(X))$ for the *image-finite part* of the projective limit of the above sequence in the category of posets with monotone maps. In other words, we take

$$\{x \in \varprojlim P_n(X) : \uparrow x \text{ is finite }\}.$$

We now prove the following, which is analogous to the property from Lemma 14:

Lemma 40. Let $g : X \to Y$ be a monotone map between posets; $h : Z \to X$ be a monotone and g-open map, where Z is image-finite. Then there exists a unique monotone and r_g-open map such that the triangle in Figure 4 commutes.

$$\begin{tikzcd} Z \arrow[rr, "h'"] \arrow[dr, "h"'] & & P_g(X) \arrow[dl, "r"] \\ & X & \end{tikzcd}$$

Figure 4: Commuting Triangle of Posets

Proof. The arguments will all be the same except we now need to show that given $a \in Z$, $h'(a)$ is a finite subset; but since we assume that $\uparrow a$ is finite, this immediately follows. ∎

Using this we can show that the poset $P^g_G(X)$ will be terminal in the desired way. The next Proposition contains the crux of the arguments:

Proposition 41. Given a poset X, $P^g_G(X)$ has the following universal property: whenever $p : Z \to X$ is a monotone map from an image-finite poset Z, there is a unique extension of p to a p-morphism $\overline{p} : Z \to P_G(X)$.

Proof. Using Lemma 40 repeatedly we construct a map $f : Z \to \varprojlim P_n(X)$. We will show that such a map is in fact a p-morphism; if we do that, then since Z is image-finite, it will factor through the image-finite part of the projective limit, and hence will give us the desired map.

So assume that $x \in Z$, and suppose that

$$f(x) \leqslant y$$

Note that by construction, $y = (a_0, a_1, ...)$ for some elements, sending the root map appropriately. Now let n be arbitrary. Then consider a_{n+1}, which by definition is a subset of $p_{n+1}[\uparrow x] = \{p_n(y) : x \leqslant y\}$. That means that there is some y' such that $x \leqslant y'$ and $f(y')$ agrees with y up to the n-the position (which follows from the commutation of the triangles in the above Lemma). Since $x \in Z$ has only finitely many successors, this means that there must be a successor $x \leqslant a$, such that $f(a)$ agrees with y on arbitrarily many positions, i.e., $f(a) = y$. This shows the result as desired. ∎

Using similar arguments, and similar ideas to Proposition 16, we can then show:

Proposition 42. The map $P_G : \mathbf{Pos} \to \mathbf{ImFinPos}$ is a functor; indeed it is left adjoint to the inclusion of $I : \mathbf{ImFinPos} \to \mathbf{Pos}$.

A few facts are worthy of note here: following the main ideas of [13], the above describes an adjunction which splits the adjunction between profinite Heyting algebras and the category of sets, a fact which is worthy of note. Moreover, as noted in this work, the construction P_G given above can be thought of as a generalization of the n-universal model – indeed, if one starts with the dual poset of the free distributive lattice on n-generators, P_G will produce precisely this same model. This illustrates an interesting connection between these two well-known constructions of the free algebra – the Ghilardi/Urquhart step-by-step construction, and the Bellissima/Grigolia/Shehtman universal model – showing that they are, in some sense, dual to each other.

7 Conclusions and Further Research

In this note we have generalized Ghilardi's construction of the free Heyting algebra generated by a finite distributive lattice to the general case, and have extracted some consequences from this – namely, the fact that the category of Heyting algebras is co-distributive, and direct proofs of amalgamation of Heyting algebras deriving from the corresponding properties for distributive lattices, together with general considerations. We have also looked at how this construction fares in different settings: when considering specific subvarieties of Heyting algebras (Boolean algebras, KC-algebras, LC-algebras), where it is shown that the construction can be reasonably adapted, and sometimes corresponds to natural modifications of the Ghilardi functor; and also we considered the relationship between the category of image-finite posets with p-morphisms and the category of posets with monotone maps, showing that reasonable modifications also yield an adjunction here.

Several questions are raised by the above considerations, both of a technical and of a conceptual nature. An immediate application which we intend to develop in a companion paper to this one is that the above adjunctions can be used to provide a coalgebraic semantics for some intuitionistic modal logics; this is done by showing that essentially the coalgebraic representation which works for distributive lattices can be "reflected" to Heyting algebras by using the V_G and P_G constructions.

Looking futher, in the case of finite distributive lattices, Ghilardi managed to show using his construction that free finitely generated Heyting algebras are bi-Heyting algebras; this relies on the fact that applying $V_g(-)$ to a finite poset yields a finite poset, which has a natural structure as a bi-Esakia space. It is not clear that even starting with a bi-Esakia space X, $V_g(X)$ will be bi-Esakia; we have been able to show that $V_r(X)$ will be co-Esakia, but all other facts remain open.

Following in the study of subvarieties of Heyting algebras, a systematic study connecting correspondence of axioms with Kripke semantics, and the appropriate modifications made to the V_G functor, seems to be in order. This does not seem immediately straightforward, as our example with KC algebras illustrates, but it might be possible to obtain more general results.

It would also be interesting to study similar constructions to the ones presented here for other signatures; a natural extension would be to study bi-Heyting algebras and bi-Esakia spaces. We expect that this should provide a fair challenge, since rather than simply adding relative pseudocomplements, such a construction would need to also add relative supplements, and even in the finite case, no such description appears to be available in the literature.

Finally, the present methods show that the category of image-finite posets includes into all posets through an inclusion admitting a right adjoint, but this explains nothing of the inclusion

$$\mathbf{Pos}_p \to \mathbf{Pos}.$$

The former category seems even less well-behaved than the category of image-finite posets, and seems to be connected with the category of Topological spaces with open and continuous maps. We leave it for further research to study how such adjunctions operate, and their consequences for the study of categories of logic, of algebras, their connections with point-free topology, and other related fields.

8 Acknowledgements

I would like to thank Silvio Ghilardi, Nick Bezhanishvili and Lingyuan Ye for several inspiring conversations and suggestions on parts of this work. I would also like to especially thank Nick Bezhanishvili for several suggestions of improvement on earlier versions of this work. I especially want to thank Mamuka Jibladze for many enthusiastic discussions on the main construction of this paper, and for his key idea in proving a lemma above.

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9 Appendix A - Raney Duality

Following [5], we refer generically to the duality between complete and completely join-prime generated algebras as *Raney duality*, due to the origins of such a duality in the work of Raney [25]; however, as pointed out in [5], the morphism part of such a duality is difficult to find in the literature. The restriction of such a duality to Heyting algebras and their corresponding maps was discussed in [20]. We refer the reader to the discussion in [22], where these and other results of a similar nature are outlined.

In this appendix we provide in full details the duality between these categories, and recall the restrictions of such a duality to the case of profinite Heyting algebras, as discussed in [4].

Definition 43. Let D be a distributive lattice. We say that $x \in D$ is *completley join-prime* if whenever $x \leqslant \bigvee_{i\in I} a_i$ for $(a_i)_{i\in I} \subseteq D$, then $x \leqslant a_i$ for some $i \in I$. We say that D is *completely join-prime generated* if for each $b \in D$, there exists a family $(c_i)_{i\in I}$ of join-prime elements such that $b = \bigvee_{i\in I} c_i$. We denote by **CCJDL** the category of complete and completely join-prime generated distributive lattices, with complete distributive lattice homomorphisms.

Let $\mathsf{Up}(-) : \mathbf{Pos} \to \mathbf{CCJDL}^{op}$ be the assignment which, given a poset P, maps it to the set of its upsets, and given a monotone map $p : P \to Q$, maps it to p^{-1}. Then we have:

Proposition 44. The map $\mathsf{Up}(-) : \mathbf{Pos} \to \mathbf{CCJDL}^{op}$ is a well-defined functor.

Proof. It is easy to check that $\mathsf{Up}(P)$ is always a complete distributive lattice with union and intersection. It is also completely join-prime generated, since the sets of the form $\uparrow x$ are easily seen to be completely join-prime. The fact that p^{-1} will be a complete distributive lattice homomorphism is likewise standard. ∎

In the opposite direction we consider the map $\mathcal{J}^{\infty}(-) : \mathbf{CCJDL} \to \mathbf{Pos}$ which, given a complete and completely join-prime generated algebra $\mathcal{D}$, assigns it the poset $(\mathcal{J}^{\infty}(\mathcal{D}), \preceq)$ of completely join-prime filters (i.e., principal filters generated by a completely join-prime element); and sends complete distributive lattice homomorphisms $h : \mathcal{D} \to \mathcal{D}'$ to the monotone map h^{-1}. Then we have:

Proposition 45. The map $\mathcal{J}^{\infty}(-) : \mathbf{CCJDL} \to \mathbf{Pos}$ is a well-defined functor.

Proof. We check that h^{-1} is monotone. Assume that $x, y \in \mathcal{J}^{\infty}(\mathcal{D}')$, and $x \preceq y$. Assume that $a \in h^{-1}[x]$. Then $h(a) \in x$, so $h(a) \in y$, so $a \in h^{-1}[y]$. This shows the desired result. ∎

The key facts establishing the duality are the following naturality requirements:

Proposition 46. The assignment:

$$\varepsilon : P \to \mathcal{J}^{\infty}(\mathsf{Up}(P))$$

given by $\varepsilon(x) = \uparrow(\uparrow x)$ is a poset isomorphism; and the assignment:

$$\eta : \mathcal{D} \to \mathsf{Up}(\mathcal{J}^{\infty}(\mathcal{D}))$$

Given by $\eta(x) = \bigcup_{i\in I}\{\uparrow a_i : a_i \text{ is completely join-prime, and } a_i \leqslant x\}$, is a distributive lattice isomorphism. Moreover, both of these assignments are natural, i.e., they witness a dual equivalence of categories.

Proof. The naturality of these assignments is easy to see. We show that both of them are isomorphisms.

To see that ε is injective, note that if $x \neq y$ then without loss of generality, $x \nleqslant y$, so $\uparrow y \nsubseteq \uparrow x$, so $\uparrow x \notin \varepsilon(y)$, so $\varepsilon(x) \neq \varepsilon(y)$. Moreover, we have that $x \leqslant y$ if and only if $\uparrow y \subseteq \uparrow x$ if and only if $\uparrow(\uparrow x) \leqslant \uparrow(\uparrow y)$. To see surjectivity, note that if F is a completely join-prime filter, then it is generated by an element $x \in P$.

Now assume that $x, y \in \mathcal{D}$ and $x \neq y$. Since $\mathcal{D}$ is completely join-prime generated, then there is some completely join-prime element $a \leqslant x$ such that $a \nleqslant y$. So $\eta(x) \neq \eta(y)$. To see that this is a complete distributive lattice homomorphism, note that

$$\eta(\bigwedge_{i\in I} a_i) = \bigcup_{j\in J}\{\uparrow b_j : b_j \leqslant \bigwedge a_i\} = \bigcap_{i\in I}\bigcup_{j\in J_i}\{\uparrow b_j : b_j \leqslant a_i\};$$

and similar for the join. Finally, to see that this is surjective, first note that if a is completely prime in $\mathcal{D}$, then $\eta(a) = \uparrow(\uparrow a)$; assume that U is an upset of $\mathcal{J}^{\infty}(\mathcal{D})$; for each $a \in U$ let $a' \in \mathcal{D}$ be the completely join-prime element which generates it. Then look at

$$b := \bigvee_{a\in U} a',$$

which exists in $\mathcal{D}$ by completeness. We claim that $\eta(b) = U$. Indeed, by the fact just proved

$$\eta(b) = \bigcup_{i \in I} \eta(a') = \bigcup_{i \in I} \uparrow(\uparrow a') = U,$$

as desired. ■

Of interest to us is the restriction of the above duality to the subcategories $\mathbf{Pos}_p$, on one hand, and **CCJHA**, the category of complete and completely join-prime generated Heyting algebras with Heyting homomorphisms, in another. We note that in both cases, these are wide subcategories - they contain all the same objects, but fewer morphisms: this is because every complete and completely join-prime generated distributive lattice is already a Heyting algebra in a canonical way: given U, V upsets we define the relative pseudocomplement as

$$U \to V := \{x : \forall y(x \leqslant y \wedge y \in U \implies y \in V)\}$$

The transformations, on the level of objects, are thus the same, and we focus on the case of morphisms:

Proposition 47. If $f : P \to Q$ is a p-morphism, then $f^{-1} : \mathsf{Up}(Q) \to \mathsf{Up}(P)$ is a complete Heyting homomorphism; if $f : \mathcal{H} \to \mathcal{H}'$ is a complete Heyting homomorphism, then $f^{-1} : \mathcal{J}^{\infty}(\mathcal{H}') \to \mathcal{J}^{\infty}(\mathcal{H})$ is a p-morphism.

Proof. The first fact follows by the usual arguments of Esakia and Jonsson-Tarski duality, and does not present any trouble. To see the latter fact, assume that $x \in \mathcal{J}^{\infty}(\mathcal{H}')$ and $y \in \mathcal{J}^{\infty}(\mathcal{H})$ and

$$f^{-1}[x] \subseteq y.$$

Let x' be the principal generator of the filter x. Consider the following sets:

$$Im(y) = \{f(a) : a \in y\} \text{ and } CoIm(y) = \{f(b) : b \notin y\}.$$

Then first we claim that

$$x' \wedge \bigwedge Im(y) \nleqslant \bigvee CoIm(y)$$

For suppose it was. Then

$$x' \leqslant \bigwedge\{f(a) : a \in y\} \to \bigvee\{f(b) : b \notin y\};$$

since f is a complete Heyting homomorphism, we have that

$$x' \leqslant f(\bigwedge\{a : a \in y\} \to \bigvee\{b : b \notin y\}).$$

So $f(\bigwedge\{a : a \in y\} \to \bigvee\{b : b \notin y\}) \in x$, so $\bigwedge\{a : a \in y\} \to \bigvee\{b : b \notin y\} \in f^{-1}[x]$, and by assumption, it is also in y. Since $\bigwedge\{a : a \in y\} \in y$ as well, then this means that $\bigvee\{b : b \notin y\} \in y$; but since y is completely prime, this is a contradiction.

Hence, by complete join-prime generation, we have that there must be a completely join-prime element c' such that $c' \leqslant x' \wedge \bigwedge\{f(a) : a \in y\}$ and $c \nleqslant \bigvee\{f(b) : b \notin y\}$. Let $c := \uparrow c'$. Then clearly $x \leqslant c$; moreover, we have that

$$f^{-1}[c] = y;$$

indeed if $a \in y$, then $f(a) \in c$; and if $b \notin y$, then $f(b) \notin c$, by construction. This shows that f^{-1} is a p-morphism as desired. ■

We also recall the main result from [4] (see Theorem 3.6). We present here a simplified proof of this duality, which avoids the detour through Esakia duality, and which follows our categorical considerations. For it, we need the following simple lemma:

Lemma 48. Let P be a poset. Then P is a directed colimit of finite posets through a diagram consisting only of p-morphisms if and only if P is an image-finite poset.

Proof. Recall that directed colimits in the category of posets consist of disjoint unions with identification along the images. Hence, if P is an image-finite poset, then we can certainly represent it as the directed colimit of its finite upsets, all of which will be finite posets as well.

Now assume that P can be represented in this way. Suppose that $x \in P$; then by definition, x must have been introduced at some stage, say P', such that $g : P' \to P$ is a p-morphism. Now since P' is finite, this then implies that $\uparrow x$ must be finite as well, otherwise the back condition could not be satisfied on g. ■

We will also help ourselves to the following lemma (see [6, Lemma 2.6]:

Lemma 49. Let $\mathcal{H}$ be a profinite Heyting algebra. Then $\mathcal{H}$ is complete and completely join-prime generated.

With this we can prove the following:

Lemma 50. Let $\mathcal{H}$ be a complete and completely join-prime generated Heyting algebra. Then for each $\mathcal{F}$ a finite Heyting algebra, it is a quotient of $\mathcal{H}$ if and only if it is a complete quotient of $\mathcal{H}$.

Proof. It is a standard fact that quotients correspond to filters, and that complete quotients correspond to complete filters. Complete filters in complete algebras must always be principal; so in the dual Esakia space, they are represented by closed upsets of the form $\varphi(a)$. Hence complete quotients into finite algebras correspond, dually, to finite upsets of the form $\varphi(a)$.

On the other hand, a finite quotient of such an algebra also corresponds to a filter, and via duality, to a finite closed upset. Let U be such a subset in the Esakia space. Note that we have that

$$U = \bigcap_{y \notin U} U_y$$

where U_y are some closed upsets. Since the algebra is complete, we have that $U' = int(\bigcap_{y \notin U} U_y)$ is a clopen upset, and clearly a subset of the former. Since the algebra is completely join-prime generated, we can prove by induction that each $\uparrow x$ in such an algebra must in fact be completely join-prime generated (by first considering the singletons, and then showing inductively that each element of the form $\uparrow x$ must be clopen as well). But this then implies that $U = U'$, i.e., that U is a clopen upset, and hence, that such finite quotients correspond to complete finite quotients, as desired. ■

We now obtain the following:

Theorem 51. *The equivalence between* $\mathbf{Pos}_p$ *and* **CCJHA** *restricts to an equivalence between* **ImFinPos** *and* **ProHA**.

Proof. By definition, a Heyting algebra is profinite if and only if it is an inverse limit of the finite Heyting algebras which are its finite quotients, through Heyting surjections. Note that since these algebras are all complete, by Lemmas 49 and 50, the inverse limit is taken in the category **CCJDL**. Hence using the above duality, a poset P is dual to a profinite Heyting algebra if and only if it is a directed colimit of its finite upsets, through maps which are p-morphisms. But by Lemma 48, we have that these are exactly the image-finite posets. ■