Inductive Rule Classes of Gödel Algebras

Rodrigo Nicolau Almeida – ILLC-UvA
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What are Π_2-rules?

Motivation

Gabbay (1981): axiomatising irreflexive frames. Idea: introduce the following rule, for p not occuring in ϕ:

$$\neg(p \to \Diamond p) \to \phi \vdash \phi.$$

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- Use it to study these rules for Gödel Algebras.

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- Use it to study these rules for Gödel Algebras.

Will focus on single-conclusion consequence relations.

Defining a Π_2-rule

$\mathcal{L}$ is a modal/intuitionistic logic of your preference (for simplicity).

Definition

A Π_2-rule is a triple (Γ, F, ϕ) such that:

1. $\Gamma \cup \{\phi\}$ is a set of $\mathcal{L}$-formulas;
2. F is a (possibly empty) set of propositional variables occurring in Γ but not in ϕ.

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Suggestively:

$$\forall \overline{p}_{\overline{p} \in F} \Gamma \vdash \phi.$$

Definition

If $(\mathcal{A}, v)$ is an algebraic model, we write $(\mathcal{A}, v) \vDash \forall \overline{p}_{\overline{p} \in F} \Gamma \vdash \phi$ to mean: if for all valuations v' differing from v at most in F, $v'(\psi) = 1$ for each $\psi \in \Gamma$, then $v(\phi) = 1$.

Π_2-rule Systems and Inductive Rule Classes

Π_2-rule systems

One can develop the notion of a "Π_2-rule system" by extrapolating from the standard case. We say that Σ is such if:

- (Monotonicity) $\forall \overline{p}_{p\in F}\Gamma \vdash \phi \in \Sigma$ then for any finite Γ' and S, we have $\forall \overline{p}_{p\in S\cup F}(\Gamma, \Gamma') \vdash \phi \in \Sigma$;
- (Bound Structurality) if $\forall \overline{p}_{p\in F}\Gamma \vdash \psi \in \Sigma$ and σ is a substitution leaving all variables in F fixed, and such that p does not occur in $\sigma(q)$ for $q \notin F$, then $\forall \overline{p}_{p\in F}\sigma[\Gamma] \vdash \sigma(\psi) \in \Sigma$
- (Renaming) If $\forall \overline{p}, q_{p\in F}\Gamma \vdash \phi \in \Sigma$, then if Γ' is Γ with all instances of q replaced by r, $\forall \overline{p}, r_{p\in F}\Gamma' \vdash \phi \in \Sigma$.

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- (Strong Reflexivity) For all sets Γ, and F a set of propositional letters, and $p_0, ..., p_n \in F$, and $\phi \in \Gamma$, we have $\forall\overline{q}_{q\in F}\Gamma \vdash \phi[\overline{\psi}/\overline{p}] \in \Sigma$, for ψ_i formulas not containing any variables from F.
- (Rule Cut) If $F = \{p_0, ..., p_n\}$ and $G = \{q_0, ..., q_k\}$ and $\Gamma(\overline{p}, \overline{r})$ is a collection of formulas, and $\forall\overline{p}_{p\in F}\Gamma \vdash \mu_i(\overline{q}, \overline{r}) \in \Sigma$ where $\Delta = \{\mu_i(\overline{q}, \overline{r}) : i \leq n\}$ is a finite set of formulas; and $\forall\overline{q}_{q\in G}\Delta \vdash \phi \in \Sigma$ is a rule, where no variable in G appears free in Γ, then $\forall\overline{p}_{p\in F}\Gamma \vdash \phi \in \Sigma$.

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Using a Lindenbaum-Tarski style argument we can prove:

Theorem (Completeness Theorem for Inductive rules)

Let Σ be a Π_2-rule system. Suppose that $\Gamma/^2\phi \notin \Sigma$. Then there is some algebra H, such that $\mathcal{H} \vDash \Sigma$, and $\mathcal{H} \nvDash \Gamma/^2\phi$.

This raises the question: what is a "variety" or a "quasivariety" in this setting?

Definition
Given two algebras $\mathcal{A} \leq \mathcal{B}$, we say that this is a $\forall$-subalgebra if for each equation $\phi(\overline{x}, \overline{y})$ in the language $\mathcal{L}$ and $\overline{a} \in \mathcal{A}$:

$$\mathcal{A} \vDash \forall \overline{x} \phi(\overline{x}, \overline{a}) \implies \mathcal{B} \vDash \forall \overline{x} \phi(\overline{x}, \overline{a}).$$

A class K of algebras is called an inductive class if it is closed under ultraproducts, products and $\forall$-subalgebras.

For example: any subdirect product is a $\forall$-subalgebra.

With some minimal adaptations from Mal'tsev's theorem we get:

Theorem

Let K *be a class of algebras. Then the following are equivalent:*

1. K *is an inductive rule class.*
2. K *is axiomatised by (∀∃-Special Horn) first-order formulas of the form:*

$$\forall \overline{x}(\forall \overline{y}(\bigwedge_{i=1}^{n} \phi_i(\overline{x}, \overline{y})) \to \psi(\overline{x}))$$

3. K *is* $\mathbb{I}\mathbb{S}_{\forall}\mathbb{P}_U\mathbb{P}^{fin}(\mathsf{K}')$ *for some class of algebras* K′.

Using these results, we can prove a basic algebraic completeness result:

Corollary

There is a dual isomorphism, Ind, between the lattice of Π_2*-rule systems, and the lattice of inductive rule classes of* $\mathcal{L}$*-algebras.*

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Upshot: We can use tools of universal algebra and algebraic logic to look at the structure of Π_2-rules and logically interesting questions herein.

Algebraic Completeness, Completed

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Upshot: We can use tools of universal algebra and algebraic logic to look at the structure of Π_2-rules and logically interesting questions herein.

As a case study we concentrate on Gödel algebras:

$$\mathsf{LC} := \mathsf{IPC} \oplus (p \to q \vee q \to p).$$

Example

The *density rule* of Takeuti and Titani:

$$\forall q(p \to q) \vee (q \to r) \vee c \vdash (p \to r) \vee c.$$

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Example

Given the formulas bc_n

$$bc_n = p_0 \vee p_0 \to p_1 \vee \ldots \vee p_0 \wedge \ldots \wedge p_{n-1} \to p_n.$$

We can consider the rule $\forall p_0, \ldots, p_n bc_n \vdash \bot$.
This rule works like an anti-axiom: it corresponds to the dual Kripke frame having at least $n + 1$ points.

Inductive Classes of Gödel Algebras

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1. (Gödel, 1933, Dummett, 1959) – The lattice of subvarieties of Gödel algebras is countable with order type $\omega + 1$.
2. (Dzik & Wronski, 1973) – Every subquasivariety of Gödel algebras is already a variety.
3. (Beckmann, Goldstern, Preining, 2008) – There are countably many first-order Gödel logics.
4. (Baasz, 1998) – On the other hand, when looking at potentially infinitary systems of Gödel logic, there are continuum many such systems.

Inductive Classes of Gödel Algebras

What about inductive classes?

Let $X = \{[n] : n \in \omega\}$ be a set containing all n-element chains.

Let λ_n be the equation defining the variety generated by the n-element chain. Then for each chain $[m]$, $[m] \vDash \lambda_n$ if and only if $m \leq n$.

Theorem

For each subset $Y \subseteq X$, $\mathbb{IR}(Y)$ forms a distinct inductive rule class. Hence there are $2^{\aleph_0}$ many inductive rule classes.

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Proof: Suppose that $Y \neq Z$. Let $[n] \in Y \notin Z$. Assume that $[n] \in \mathbb{IR}(Z)$. Then $[n]$ is a a $\forall$-subalgebra of an ultraproduct of finite products of elements from Z (by model theoretic completeness).

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Say that $[n] \leq_\forall \prod_{i \in I} \mathcal{H}_i/U$; then $\prod_{i \in I} \mathcal{H}_i/U \vDash \lambda_n$ because it a $\forall$-subalgebra. By Los theorem, then for ultrafilter many i, $\mathcal{H}_i \vDash \lambda_n$. So if $\mathcal{H}_i \cong \prod_{i=1}^{n} [k_i]$ we have that $[k_i] \vDash \lambda_n$, so $k_i \leq n$. Since $[n] \notin Z$, then for $k = \max(\{k \in Z : k < n\})$ we have that $[k_i] \vDash \lambda_k$. But then $\mathcal{H}_i \vDash \lambda_k$. Since this holds for ultrafilter many i, $\prod_{i \in I} \mathcal{H}_i/U \vDash \lambda_k$, which implies that $[n] \vDash \lambda_k$, a contradiction. □

In fact, one can construct explicit rules separating, for instance [3] from [2]:

$$\forall q(\neg\neg q \to q \vee p) \vdash p.$$

This rule can be falsified in $\mathbf{2}$, by taking $p = 0$, but it cannot fail in $\mathbf{3}$. The problem here seems to be that finite algebras become very separated when looking at inductive classes.

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One might then expect that the structure of such classes is hopeless. But it seems it is still possible to say interesting things.

Some Final Remarks

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More precisely we can even characterise all extensions of LC which are "inductively structurally complete" – where all Π_2-rules which are admissible are derivable:

Theorem
An inductive rule class is inductively structurally complete if and only if it is of the form $\mathbb{IR}([n])$ *or* $\mathbb{IR}([\mathbb{Q}])$.

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This turns out to depend on a nice structure theorem for algebras and ∀-subalgebras.

Future Work

Some natural questions follow, both in the seting of LC and elsewhere:

- How do model completions interact with these rules?
- What can be said in general about admissibility of Π_2-rules?
- How do these rules relate to other interesting phenomena like implicit connectives?

Thank you!

Questions?