Rules in Algebraic and Geometric Form: Structural Completeness of bi-Intuitionistic Logic

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I will assume a passing familiarity with intuitionistic logic, and sometimes more than that.

What is a rule system?

Fix a given logical language $\mathcal{L}$; we will be general and somewhat sketchy.

Definition

Let $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$. We say that $\vdash$ is a *rule system* if it satisfies, for every $\Gamma, \Delta \subseteq \mathcal{L}$ and $\psi, \phi \in \mathcal{L}$:

1. (Reflexity) $\phi \vdash \phi$
2. (Cut Rule) $\Gamma \vdash \phi$ for every $\phi \in \Delta$, and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.
3. (Substitution Invariance) if $\Gamma \vdash \phi$ and σ is a substitution, then $\sigma[\Gamma] \vdash \sigma(\phi)$.

Example

Consider IPC or CPC with $\vdash$ standing for usual (Hilbert-style) derivability.

Admissible Rules and Structural Completeness

Definition
Given a rule system $\vdash$, we say that a rule Γ/ϕ is

- **Admissible** if for each substitution σ, whenever $\vdash \sigma(\psi_i)$ for each $\psi_i \in \Gamma$, then $\vdash \sigma(\phi)$.
- **Derivable** if it holds that $\Gamma \vdash \phi$.

We say that a rule system $\vdash$ is *structurally complete* if all admissible rules are derivable.

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Example
CPC is structurally complete. An example of an admissible rule over IPC which is not derivable is:

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

To each rule system $\vdash$ we associate a *quasivariety*.

Definition

A class K of algebras is called a quasivariety if it is closed under subalgebras, products and ultraproducts; a quasivariety is called a variety if it is closed under homomorphisms.

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It is possible to pin structural completeness down to universal algebra:

Proposition

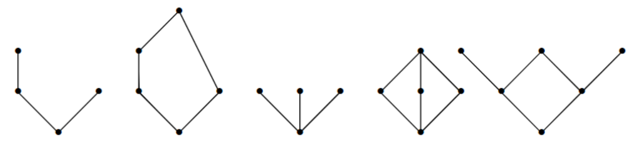
(Bergman, 1988) The following are equivalent for a rule system $\vdash$ with an associated quasivariety K:

1. $\vdash$ *is structurally complete;*
2. *Every proper subquasivariety of K generates a proper variety than the one generated by K.*

Relational Understanding

If we are working in a setting like IPC we have a duality between algebraic models and relational models. For our purposes we have that:

$$\begin{aligned} \text{Finite Heyting Algebras} &\rightsquigarrow \text{Finite Posets} \\ \text{Subalgebras} &\rightsquigarrow \text{P-morphic Image.} \\ \text{Finite Products} &\rightsquigarrow \text{Disjoint Unions.} \end{aligned}$$



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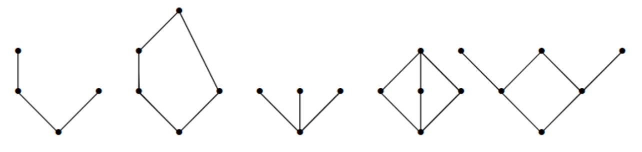

Figure 1: Some posets

Some very strong results can be shown using these tools:

Theorem (Citkin,1978)
If L is a superintuitionistic logic, then L is hereditarily structurally complete (with respect to logics) if and only if L does not have the above five posets as models.

The former holds for intuitionistic logic. But there is an extension of intuitionistic logic, which adds a new connective.

Definition
Consider the language $\mathcal{L}$ of IPC, and enrich it with a connective "$-$". This connective has the following intended semantics over intuitionistic models $(\mathfrak{M}, \leq, V)$:

$$\mathfrak{M}, x \Vdash \phi - \psi \iff \exists y \leq x (\mathfrak{M}, y \Vdash \phi \text{ and } \mathfrak{M}, y \nVdash \psi)$$

The logic bi-IPC is the logic of these structures.

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The algebraic models of these logics are bi-Heyting algebras, and there is also a duality between (finite) bi-Heyting algebras and (finite) posets, but now with p-morphisms that also look back.

My PhD supervisor asked me a few months ago: “Could we have the kind of characterisation offered by Citkin, for bi-Heyting algebras?”.

The story is not complete, but it starts with the following fact:

Proposition

(Kowalski et.al, 2020) If K *is a variety of bi-Heyting algebras, and* K *is not the variety of Boolean algebras, then* K *contains the three element chain.*

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And from this we can obtain the following:

Proposition
There are no structurally complete bi-IPC logics except for classical logic.

Theorem
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Assume that K is any variety which is not Boolean. Hence by the above proposition, $\mathbf{3} \in \mathsf{K}$. Now consider

$$\mathbb{Q}(\{\mathcal{B} : \mathcal{B} \cong \mathcal{H} \times \mathbf{2}, \mathcal{H} \in \mathsf{K}\}).$$

Clearly the variety generated by this quasivariety is not proper. But the quasivariety itself is proper, since we claim that $\mathbf{3}$ is not there. To see this, note that if it were, since it is finite it would belong to

$$\mathbb{SP}^{fin}(\{\mathcal{B} : \mathcal{B} \cong \mathcal{H} \times \mathbf{2}, \mathcal{H} \in \mathsf{K}\})$$

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Structural Completeness in bi-IPC (continued)

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Thinking dually we can see that this is a contradiction: if X is any such space, it will be isomorphic to $Y \sqcup \{\bullet\}$ for some Y; but then the loose point can be mapped to nowhere in $\mathbf{3}$. This shows the quasivariety is proper, and so the logic is not structurally complete. □

Some facts can never the less be shown (I can discuss them if there is time):

Theorem

The logic bi-LC *is hereditarily actively structurally complete.*

Questions I'm Currently analysing

- Can we characterise all actively structurally complete (quasi)varieties of bi-intuitionistic logics?
- Can we extend the characterization for intuitionistic logics to rule systems?
- Can we find an admissible basis of rules for bi-IPC?
- Can we describe how structural completeness is distributed, in relationship with other strong metalogical properties? (Craig Interpolation, Uniform Interpolation, Hereditary FMP, etc)