Colimits of Heyting Algebras Through Esakia Duality

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Free Algebras of Logic

Motivation

Free Boolean algebras are very well-understood: for a set X they can be described as the duals of

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Having a good grasp of free algebras is intimately related to having a good grasp of colimits of these algebras. Logically, this has several natural desirable applications: interpolation, conservativity, etc.

Definition

An algebra $\mathcal{H} = (H, \wedge, \vee, \to, 0, 1)$ is called a Heyting algebra if $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice satisfying for every a, b, c:

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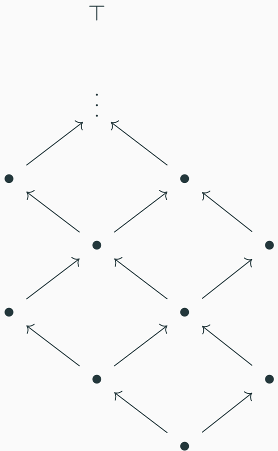

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In this talk we present a construction which gives us access to free Heyting algebras, coproducts of Heyting algebras, and related constructions.

Generating Heyting Algebras

Heyting Algebras from Distributive Lattices

Describing the freely generated algebras amounts to studying the adjunction:

$$\mathsf{HA} \underset{\textit{Forget}}{\overset{\textit{Free}}{\rightleftarrows}} \mathsf{Set}$$

Figure 2: Free-Forgetful Adjunction

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Instead of studying it directly, we "split" this adjunction:

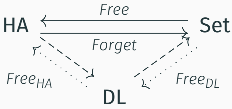

Figure 3: Split Free Forgetful Adjunction

Key Intuition: We think of generating Heyting algebras as adding infinitely many layers of implications to a distributive lattice.

Heyting Algebras from Distributive Lattices, algebraically

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Let $\mathcal{D}_0 = \mathcal{D}$. Then set

$$\mathcal{D}_1 = F_{DL}(\{a \to b : a, b \in D\})/\Theta$$

where Θ contains:

1. Axioms of Heyting algebras;
2. Axioms enforcing elements of the form $1 \to a$, for $a \in D$, to behave like the elements from D.

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We set the map $i_1 : \mathcal{D}_0 \to \mathcal{D}_1$ sending a to $[1 \to a]$, which is a homomorphism by force. We then iterate; but for $\mathcal{D}_2$ we need to also add one more rule to Θ:

3. Axioms ensuring that if $a, b \in D$, then:

$$1 \to_{D_2} (a \to_{D_1} b) \equiv (1 \to_{D_1} a) \to_{D_2} (1 \to_{D_2} b).$$

This is how we define $\mathcal{D}_2$, and i_2 is defined similarly.

We then iterate infinitely:

1. $\mathcal{D}_{n+1} = F_{DL}(\{a \to b : a, b \in D_n\})/\Theta$, where Θ is defined as above;
2. $i_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$ sends a to $[1 \to a]$.

Problem: We would like to say that the free Heyting algebra generated by $\mathcal{D}$ is the union of all of these. But we do not know whether this is a chain of embeddings, or really anything about this construction.

Dual Perspectives

To clarify the properties of the above construction, we analyze it dually.

One-step dual constructions

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The idea of dual one-step constructions is not-new, and well-studied in Modal Logic. In the case where we generate from finite Heyting algebras, this was studied by Ghilardi (1992), generalizing previous work of Urquhart (1973); Bezhanishvili and Gehrke (2009) gave a detailed outline of this method for various classes.

Vietoris Spaces

Let $(X, \leq, \tau)$ be a Priestley space (compact, totally order-disconnected space). We denote by $(V(X), \supseteq)$ the Vietoris Hyperspace of X with reverse inclusion.

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$V(X)$ is the set of closed subsets of X; it has a basis consisting of sets:

$$[U] = \{C \in V(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in V(X) : C \cap V \neq \emptyset\}$$

where $U, V \in \text{Clop}(X)$.

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We write $V_r(X)$ for the set of closed and rooted subsets of X.

The following is a crucial lemma:

Lemma
The space $V_r(X)$ is a closed subspace of $V(X)$.

Proof.
Consider $V(X) \times X$. Then look at the following spaces:

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One can show that both of them are closed subspaces of $V(X) \times X$. Their intersection is thus closed. But $V_r(X)$ is the projection on the first coordinate of such a space – and continuous functions between Stone spaces are closed maps. □

Definition
Let X, Y, Z be Priestley spaces, and $g : X \to Y$ and $f : Z \to X$ be Priestley morphisms. We say that f is open relative to g (g-open for short) if it satisfies the following:

$$\forall a \in Z, \forall b \in X, (f(a) \leq b \implies \exists a' \in Z, (a \leq a' \ \& \ g(f(a')) = g(b)). \tag{*}$$

Given $S \subseteq X$ a closed subset, we say that S is g-open (understood as a poset with the restricted partial order relation) if the inclusion is itself g-open.

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Lemma
A map $f : Z \to X$ as above is g-open if and only if whenever $U, V \subseteq Y$ are clopen upsets, ${\downarrow}g^{-1}[U] - g^{-1}[V]$ is clopen in X, and:

$$f^{-1}[X - {\downarrow}(g^{-1}[U] - g^{-1}[V])] = X - {\downarrow}(f^{-1}g^{-1}[U] - f^{-1}g^{-1}[V]).$$

i.e., f^{-1} preserves relative pseudocomplements indexed by g^{-1}.

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Let X, Y be a Priestley spaces, $g : X \to Y$ Priestley morphism. We denote by $V_r(X)$ the set of closed and rooted subsets of X. We denote by $V_g(X)$ the set of closed, rooted and g-open subsets of X.

Then we also have:

Lemma
Let X, Y be Priestley spaces, $g : X \to Y$ be a Priestley morphism such that X whenever $U, V \subseteq Y$ are clopen, ${\downarrow}(g^{-1}[U] - g^{-1}[V])$ is clopen in X. Then $V_g(X)$ is a closed subspace of $V_r(X)$.

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Proposition
If X, Y are Priestley spaces such that $g : X \to Y$ is a Priestley morphism in the above conditions, then $V_g(X)$ is a Priestley space. Moreover, the map $r_g : V_g(X) \to X$ sending each rooted subset C to its root (the "root map") is a continuous and order-preserving surjection which is g-open.

We can now proceed with the main construction:

Definition

Let $g : X \to Y$ be a Priestley morphism. The g-Vietoris complex over X $(V^g_\bullet(X), \leq_\bullet)$, is a sequence

$$(V_0(X), V_1(X), ..., V_n(X))$$

connected by morphisms $r_i : V_{i+1}(X) \to V_i(X)$ such that:

1. $V_0(X) = X$;
2. $r_0 = g$
3. For $i \geq 0$, $V_{i+1}(X) := V_{r_i}(V_i(X))$;
4. $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ is the root map.

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We denote the projective limit of this family by $V^g_G(X)$, and omit g when this is the terminal map to $\mathbf{1}$.

It is not hard to show that $V^g_G(X)$ is always an Esakia space.

Then we can prove:

Proposition

Let $g : X \to Y$ be a Priestley morphism, and let $k : Z \to X$ be a g-open Priestley morphism. Then there exists a unique extension of k to a p-morphism $\tilde{k} : Z \to V_G^g(X)$.

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By considering maps appropriately, one can use this to prove:

Corollary

The assignment V_G is a functor mapping **Pries** *of Priestley spaces and Priestley morphisms, to the category* **Esa** *of Esakia spaces and p-morphisms. It is the right adjoint of the inclusion.*

Some Applications

Using the above results we can obtain a more concrete description of free Heyting algebras:

Corollary

Let S be any set. Then the Esakia space $V_G(2^S)$ is dual to the free Heyting algebra on S-generators.

In the finite case this recovers Urquhart/Ghilardi's construction of the free n-generated Heyting algebra.

Coproducts of Heyting algebras

Given X, Y two Esakia spaces, their cartesian product $X \times Y$ is again an Esakia space; however this is not the categorical product.

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Let X, Y be two Esakia spaces. Then the triple

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Using this we can prove some basic facts about the category of Heyting algebras, directly:

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The category of Heyting algebras has amalgamation.

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And we can prove others which we would not know how to prove otherwise:

Theorem
The category of Heyting algebras is co-distributive.

Open Questions

There are several open questions raised by this, of a technical and conceptual nature:

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1. Is $V_g(X)$ a (bi-)Esakia space whenever X is one?
2. Can one construct free bi-Heyting algebras using some modifications of the above ideas?
3. Can this description be used for instance to study model theoretic questions about the first-order theory of (free) Heyting algebras?

Thank you!

Questions?