A Note on Uniform Interpolation of Modal Logics

Rodrigo Nicolau Almeida

Wednesday 11$^{\text{th}}$ June, 2025

1 Introduction

In this note we collect some well-known results about uniform interpolation in modal and superintuitionistic logics. All results are credited as due. We will look at two kinds of problems:

(i) *Uniform Craig interpolation*, which boils down to a definability problem in terms of adjoints of free algebras;

(ii) *Uniform deductive interpolation*, which boils down to a problem of existence of model completions.

We will tackle these problems in turn.

2 Uniform Craig Interpolation

Let $\mathcal{K}$ be a class of ordered algebras, possessing free algebras; we will say that $\mathcal{K}$ has the *uniform definability property* if whenever $[k] = \{p_1, ..., p_k\}$ then the inclusion:

$$i : \mathcal{F}_{\mathcal{K}}([k]) \hookrightarrow \mathcal{F}_{\mathcal{K}}([k+1]),$$

has a left and a right adjoint; explicitly, this means that there are two order preserving maps $\forall_i : \mathcal{F}_{\mathcal{K}}([k+1]) \to \mathcal{F}_{\mathcal{K}}([k])$ and $\exists_i : \mathcal{F}_{\mathcal{K}}([k+1]) \to \mathcal{F}_{\mathcal{K}}([k])$, such that for each $a \in \mathcal{F}_{\mathcal{K}}([k])$ and $b \in \mathcal{F}_{\mathcal{K}}([k+1])$ we have:

$$\exists_i(b) \leq a \iff b \leq i(a) \text{ and } i(a) \leq b \iff a \leq \forall_i(b).$$

Let $\mathcal{K} = \mathsf{Alg}(L)$ for a logic L. In this case, the free algebras are precisely the algebras of formulas. Then we are saying that for each formula $\phi(\overline{p}, q)$, there are formulas $\exists q\ \phi(\overline{p}, q)$ and $\forall q\ \phi(\overline{p}, q)$ over $\overline{p}$, with the above property.

Definition 2.1. We say that a logic L has the *uniform Craig interpolation property* if and only if given $\phi(\overline{p}, \overline{r})$, there are two formulas χ_L and χ_R in the language of $\overline{p}$, such that for all $\psi(\overline{p}, \overline{r})$, if $\vdash \phi \to \psi$ then χ_L is a (left) uniform interpolant for this sequent; and whenever $\vdash \psi \to \phi$ then χ_R is a (right) uniform interpolant for this sequent.

The first thing to observe is then the following:

Proposition 2.2. *Let L be a logic with the Craig interpolation property and the uniform definability property. Then L has the uniform Craig interpolation property.*

Proof. Exercise. ■

One thing we also note is that uniform definability is immediate for locally tabular logics:

Proposition 2.3. *Let L be a locally tabular logic. Then L has the uniform definability property.*

Proof. Let $\phi(\overline{p}, q)$ be a formula. Consider:

$$\tilde{\exists} q\ \phi(\overline{p}, q) \coloneqq \bigwedge \{\chi(\overline{p}) \ : \vdash_L \chi(\overline{p}) \to \phi(\overline{p}, q)\}.$$

Since L is locally tabular, i.e., there are only finitely many formulas over $\overline{p}$ up to equivalence, then this is a valid definition. It is easy to verify that this gives the left adjoint to the inclusion. The right adjoint is defined in a similar way. ■

The above is of course no more than the consequence of the fact that if L is locally tabular, the free finitely generated algebras are finite, and hence the adjoints exist by abstract nonsense (i.e., the adjoint functor theorems).

The purpose of the remaining sections will be to establish some basic properties of uniform definability. These properties were originally established by Ghilardi [2]; the presentation given here is a slightly modified version of these results. We will be focusing on the modal case, although the intuitionistic case can be treated in a similar fashion, with additional care necessary to account for the upwards closure of formulas.

2.1 Bisimulations and n-bisimulations

Definition 2.4. Let $\phi \in \mathcal{L}$. We define the *modal depth* of ϕ by recursion as follows:
(i) $md(p) = 0$;
(ii) $md(\phi \wedge \psi) = \max(md(\phi), md(\psi))$ and $md(\neg\phi) = md(\phi)$;
(iii) $md(\Box\phi) = md(\phi) + 1$.

Definition 2.5. Let $(\mathfrak{M}, x) = (W, R, V)$ and $(\mathfrak{N}, y) = (W', R', V')$ be two models over $\overline{p}$. We say that a relation $S_n \subseteq W \times W'$ is an *n-bisimulation* based on (x, y) if there are relations $S_n \subseteq ... \subseteq S_k \subseteq ... \subseteq S_0$ for each $0 \leq k \leq n$, such that xS_ny and:
(i) Whenever wS_0w' then $w \in V(p) \iff w' \in V'(p)$ for each $p \in \overline{p}$;
(ii) For each $k < n$, whenever $wS_{k+1}w'$ and wRv there is some $w'Rv'$ such that vS_kv'.
(iii) For each $k < n$, whenever $wS_{k+1}w'$ and $w'Rv'$ there is some wRv such that vS_kv'.

We write $\mathfrak{M}, x \leftrightarrows_n \mathfrak{N}, y$ to mean that there is an n-bisimulation between the two models. We write $\mathfrak{M}, x \leftrightarrows \mathfrak{N}, y$ to mean that there is a bisimulation between two models simpliciter. We also recall, and prove, the following basic result from modal logic:

Proposition 2.6. *Given two models $\mathfrak{M}, x$, and $\mathfrak{N}, y$ we have that $\mathfrak{M}, x \leftrightarrows_n \mathfrak{N}, y$ if and only if $\mathfrak{M}, x$ and $\mathfrak{N}, y$ satisfy the same formulas of modal depth n (resp. implication rank n).*

Proof. First suppose that $\mathfrak{M}, x \leftrightarrows_n \mathfrak{N}, y$. Then by induction its possible to show that the two models satisfy the same formulas of modal depth n.

Conversely assume that the two models satisfy the same formulas of modal depth n. Consider for each k the following relation for $w \in W$ and $w' \in W'$:

$$w \equiv_k w' \iff \Big((\forall\phi, md(\phi) \leq k)\ \mathfrak{M}, w \Vdash \phi \iff \mathfrak{N}, w' \Vdash \phi\Big).$$

Then let $S_n = \equiv_n$; we show that this is an n-bisimulation. It is obvious that S_0 is a 0-bisimulation. Now assume that $w \equiv_{k+1} w'$. Suppose that wRv. Let $T = \{\phi : md(\phi) \leq k, \mathfrak{M}, v \Vdash \phi\}$. Note that since up to equivalence there are only finitely many formulas of modal depth k, then T is essentially a finite set. Let $\chi := \bigwedge T$. Then note that $\Diamond\chi$ is a formula of modal depth $k + 1$. By assumption, since $\mathfrak{M}, w \Vdash \Diamond\chi$, then $\mathfrak{N}, w' \Vdash \Diamond\chi$; hence there is some $w'Rv'$ where $\mathfrak{N}, v' \Vdash \chi$. But then certainly $v \equiv_k v'$, since χ is the complete theory of modal depth at most k of v. This shows that $\mathfrak{M}, x \equiv_n \mathfrak{N}, y$ is an n-bisimulation, as desired. ■

Using the aforementioned Proposition we will identify formulas of this language with classes of models.

Proposition 2.7. *Let L be a logic with the finite model property. Let $\mathcal{K}$ be a class of (finite or infinite) models of L, over $\overline{p}$. Then the following are equivalent:*
(i) $\mathcal{K}$ is closed under n-bisimulation;
(ii) There is a formula ϕ of modal depth at most n such that for each $\mathfrak{M}, x$ a finite model, $\mathfrak{M}, x \Vdash \phi$ if and only if $(\mathfrak{M}, x) \in \mathcal{K}$.

Proof. This follows immediately from Proposition 2.6. ■

The key of our analysis lies thus in the fact that the logics that we are interested in have *normal forms*:

Definition 2.8. Let $n \in \omega$ and ϕ_n a formula. We say that ϕ_n is an *n-normal form* if $\mathcal{K}(\phi_n)$ is an n-bisimulation equivalence class.

In such logics, every formula is a join of finitely many normal forms, which we will make use of extensively.

2.2 Bisimulation quantifiers and the semantics of adjoints

One typical way, found often in the literature (see e.g. [2] or [6]) is as a result of "bisimulation-quantifier elimination". In this section we will explain this, and justify this characterization for several logics of interest.

Definition 2.9. Let $\mathsf{Prop} = \{p_1, ..., p_n\}$ be a set of propositional letters, and let M be a language. We call $\mathcal{L}_{M^2}(\mathsf{Prop}) = (\mathcal{L}(M), \{\exists p_i\}_{p_i \in \mathsf{Prop}})$ the language of M enriched with *bisimulation-quantifiers.*

Recall that given a Kripke model $\mathfrak{M} = (W, R, V)$, where $V : \{p_1, ..., p_n\} \to \mathcal{P}(W)$ (or $V : \{p_1, ..., p_n\} \to \mathcal{U}(W)$, when dealing with posets), and a point $x \in W$, we denote by $(\mathfrak{M}, x)$ the pointed Kripke model. Given such a model, and $S \subseteq \mathsf{Prop}$, we denote by $\mathfrak{M}^S$ the tuple (W, R, V_S) where $V_S = V \cap S^2$ is the restriction. When $S = \mathsf{Prop} - \{p_n\}$ we will denote this simply by $\mathfrak{M}^{p_n}$.

The semantics of bisimulation-quantified logics are the following:

Definition 2.10. Given a formula $\phi \in \mathcal{L}_{M^2}(\overline{p}, q)$ we give semantics to ϕ over Kripke models by:
- (i) If ϕ is a propositional letter, $\bot$ or $\top$, or the primary connective of ϕ from M, then the semantics is the usual one.
- (ii) $\mathfrak{M}, x \Vdash \tilde{\exists} p_i\ \phi$ if and only if there is a model $\mathfrak{N}, y$ over $\overline{p}$, such that $\mathfrak{M}^{p_i}, x \leftrightarroweq \mathfrak{N}, y$, and $\mathfrak{N}, y \Vdash \phi$.

Note the following:

Proposition 2.11. *For each model $(\mathfrak{M}, x)$ over $(\overline{p}, q)$, and each formula $\phi(\overline{p}, q)$ we have:*
- *(i) $\mathfrak{M}, x \Vdash \phi(\overline{p}, q) \to \tilde{\exists} q\ \phi(\overline{p}, q)$;*
- *(ii) Whenever $\mathfrak{M}, x \Vdash \tilde{\exists} q\ \phi(\overline{p}, q) \to \chi$, then $\mathfrak{M}, x \Vdash\ \phi(\overline{p}, q) \to \chi(\overline{p})$.*

Moreover, if for each finite model, $\mathfrak{M}, x \Vdash \phi \to \psi$, then for each finite model $\mathfrak{M}, x \Vdash \tilde{\exists} q\ \phi \to \tilde{\exists} q\ \psi$.

Proof. Exercise. ∎

Corollary 2.12. *Assume that L is a logic which has the FMP. Suppose that for each $\phi(\overline{p}, q)$ in the language $\mathcal{L}(M)(\overline{p}, q)$, there is a formula $\psi(\overline{q})$ in the same language such that for each finite model $\mathfrak{M}, x$:*

$$\mathfrak{M}, x \Vdash \psi(\overline{p}) \iff \mathfrak{M}, x \Vdash \tilde{\exists} q\ \phi(\overline{p}, q).$$

Then L has the uniform definability property.

Proof. Note that if ψ exists, then by Proposition 2.11, we obtain a uniform interpolant (check this!). ∎

This Corollary can be strengthened to an equivalence, so long as we make some provisions:

Theorem 2.13. *Let L be a logic with the finite model property, such that L is:*
- *(i) Compact, and such that any consistent set Γ of formulas can be satisfied in an m-saturated model[1], or;*
- *(ii) For each finite L-model $(\mathfrak{M}, x)$, there is a natural number k, such that for any finite model $(\mathfrak{N}, y)$, if $(\mathfrak{M}, x) \leftrightarroweq_k (\mathfrak{N}, y)$ then $(\mathfrak{M}, x) \leftrightarroweq (\mathfrak{N}, y)$.*

Then if L has the uniform definability property if and only if for each $\phi(\overline{p}, q)$ the formula $\exists_i(\phi)$ given by the adjoint is semantically equivalent to $\tilde{\exists} q \phi$.

Proof. One direction is Corollary 2.12. For the other, we will show that for each finite model $(\mathfrak{M}, x)$ over $\overline{p}$, we have:

$$(\mathfrak{M}, x) \Vdash \exists_i\ \phi \iff (\mathfrak{M}, x) \Vdash \tilde{\exists} q\ \phi.$$

Indeed, if $\mathfrak{M}, x \Vdash \tilde{\exists} q\ \phi$, let $(\mathfrak{M}, x)^q \leftrightarroweq (\mathfrak{N}, y)$, where $(\mathfrak{N}, y) \Vdash \phi(\overline{p}, q)$. By assumption, then $(\mathfrak{N}, y) \Vdash \exists_i \phi$, and since this is in the reduced language, then $(\mathfrak{M}, x) \Vdash \exists_i \phi$. Conversely, assume that $(\mathfrak{M}, x) \Vdash \exists_i \phi$. We consider the two cases:

[1] These hypotheses are met, for instance, by any $\mathcal{D}$-persistent logic, of which are examples all Sahlqvist logics. For the definition of an m-saturated model, see for e.g. [1, pp.92].

(i) If L satisfies (i), then consider:

$$\{\chi \in \mathcal{L}(\overline{p}) : \mathfrak{M}, x \Vdash \chi\} \cup \{\phi\}.$$

Assume that this is not consistent. Since the logic is compact, there is a single formula χ such that $\vdash_L \phi \to \neg\chi$; since this is in the reduced language, $\vdash_L \exists_i \phi \to \neg\chi$, which is a contradiction, since then $(\mathfrak{M}, x) \Vdash \neg\chi$. Hence let $(\mathfrak{N}, y)$ be an m-saturated model satisfying the above formulas. Then by the Hennessy-Milner theorem for m-saturated models [1, Proposition 2.54], noting that $(\mathfrak{M}, x)$, being finite, is always m-saturated, $(\mathfrak{M}, x) \leftrightarroweq (\mathfrak{N}, y)$. Since also $(\mathfrak{N}, y) \Vdash \phi$, then by definition, $(\mathfrak{M}, x) \Vdash \tilde{\exists} q\phi$.

(ii) If L satisfies (ii), run the same argument as in (i), except considering:

$$\{\phi\} \cup Th_k(\mathfrak{M}, x) = \{\chi \in \mathcal{L}(\overline{p}) : \mathfrak{M}, x \Vdash \chi, md(\chi) \leq k\}.$$

In both cases we obtain the result. ■

The above theorem hence applies to a large number of logics: as mentioned, any $\mathcal{D}$-persistent logic, on one hand, and several non-compact logics which are of interest: **Grz**, **K4**.**Grz**, **GL**, amongst others. For example, **Grz** has the property outlined in the above theorem, since given its finite models are finite posets, then whenever $(\mathfrak{M}, x)$ is such a model, then the cardinality of $\mathfrak{M}$ serves as a bound. We also note that the two properties are disjoint: for example, **S4** clearly has property (i), but it does not have property (ii), since a two-element cluster has arbitrarily large k-bisimilar models.

For logics without these assumptions, one still gets a semantic description of the uniform interpolant:

Definition 2.14. Given a formula $\phi(\overline{p}, q)$, we define the formula:

$$\mathfrak{M}, x \Vdash \tilde{\exists} q^{<\omega} \phi \iff \forall k\ \exists (\mathfrak{N}, y), (\mathfrak{M}, x)^p \leftrightarroweq_k (\mathfrak{N}, y), \text{ and } (\mathfrak{N}, y) \Vdash \phi.$$

Corollary 2.15. *Let L be a logic with the finite model property. Then L has the uniform definability property if and only if for each $\phi(\overline{p}, q)$ the formula $\exists_i(\phi)$ given by the adjoint is semantically equivalent to $\tilde{\exists} q^{<\omega}\phi$.*

Proof. Run the above proofs replacing full blown bisimulation by k-bisimulation in the appropriate places. ■

Definition 2.16. Let L be a logic. A logic L is said to be *uniformizable* if it satisfies either of the conditions of Theorem 2.13.

Given $\mathcal{K}$ a class of models over $(\overline{p}, q)$ let $\pi_q[\mathcal{K}]$ be the bisimulation closure of the class of reducts of models from $\mathcal{K}$ to $\overline{p}$. We thus obtain the following necessary and sufficient conditions for a logic to have uniform definability:

Theorem 2.17. *Let L be a uniformizable logic. Then L has uniform definability if and only if whenever $\mathcal{K}$ is a class of L-models which is an n-bisimulation equivalence class, then the following equivalent conditions hold:*

(i) There is some k such that $\pi_q[\mathcal{K}]$ is closed under k-bisimulation;

(ii) Whenever $(\mathfrak{M}, x) \in \pi_q[\mathcal{K}]$, so that $(\mathfrak{M}, x) \leftrightarroweq (\mathfrak{M}', x')^q$ and $(\mathfrak{M}, x) \leftrightarroweq_k (\mathfrak{N}, y)$, then there is some model $(\mathfrak{N}', y')$ over $(\overline{p}, q)$ such that $(\mathfrak{N}', y')^q \leftrightarroweq (\mathfrak{N}, y)$ and $(\mathfrak{N}', y') \leftrightarroweq_n (\mathfrak{M}', x')$.

Proof. (i) and (ii) are indeed equivalent for $\mathcal{K}$ in these conditions: it is immediate from the definitions that (ii) implies (i). To see the converse, if $(\mathfrak{M}, x) \in \pi_q[\mathcal{K}]$, hence $(\mathfrak{M}, x)^q \leftrightarroweq (\mathfrak{M}', x')$ for $(\mathfrak{M}', x')$ a reduct of a model in $\mathcal{K}$, and $(\mathfrak{M}, x) \leftrightarroweq_k (\mathfrak{N}, y)$, then by assumption of k-bisimulation closure, $(\mathfrak{N}, y) \in \pi_q[\mathcal{K}]$; hence there is some $(\mathfrak{N}', y') \in \mathcal{K}$ such that $(\mathfrak{N}', y')^q \leftrightarroweq (\mathfrak{N}, y)$. But since we take $\mathcal{K}$ as a bisimulation equivalence class, also $(\mathfrak{N}', y') \leftrightarroweq_n (\mathfrak{M}', x')$.

Now if every class $\mathcal{K}$ which is an n-bisimulation equivalence class has this property, note that if $\mathcal{K}$ is an arbitrary class closed under n-bisimulation, then $\pi_q[\mathcal{K}]$ must be closed under k-bisimulation (by taking the maximum). Hence for each formula $\phi(\overline{p}, q)$, there is a formula $\psi(\overline{p})$ satisfying the semantics of $\tilde{\exists} q\phi$, which by Theorem 2.13 means that L has uniform definability.

Conversely, let L be a logic which has uniform definability. Let ϕ be an arbitrary formula which is an n-equivalence class. Then the fact that $\tilde{\exists} q\phi$ is represented by a formula means precisely that $\pi_q[\mathcal{K}(\phi)]$ is closed under k-bisimulation for some k. ■

Note that the conditions of this theorem have a flavour very analogous to amalgamation: one has two models which share a common model, in two distinct ways, and there is some way to glue them together.

2.3 Uniform definability in modal logic K

We will now apply Theorem 2.17 to prove that **K** has the uniform definability property.

Lemma 2.18 (Combinatorial Lemma)**.** *Let* $(\mathfrak{M}, x)$ *be a finite model over* $\overline{p}$ *such that* $(\mathfrak{M}, x) \leftrightarrows (\mathfrak{M}', x')^q$*, and* $(\mathfrak{M}, x) \leftrightarrows_n (\mathfrak{N}, y)$*. Then there is some* $(\mathfrak{N}', y')$ *over* $(\overline{p}, q)$ *such that* $(\mathfrak{N}', y')^q \leftrightarrows (\mathfrak{N}, y)$ *and* $(\mathfrak{N}', y') \leftrightarrows_n (\mathfrak{M}', x')$*.*

Proof. Let $\mathfrak{M} = (M, R, V)$, $\mathfrak{M}' = (M', R', V')$ and $\mathfrak{N} = (N, R, J)$ be the three models. Fix Z a bisimulation between M and M', and $(S_n \subseteq ... \subseteq S_0)$ an n-bisimulation between M and N. Consider the set:

$$\begin{aligned} E :=&\{(a, c, b) \in M' \times M \times N : (a, c) \in Z, (c, b) \in S_k\} \\ &\{(\infty, b) : b \in N\}. \end{aligned}$$

We give this the following relation R:

(i) $(a, c, b)R(a', c', b')$ if and only if $aR'a'$, cRc', bRb', and $(c, b) \in S_{k+1}$ and $(c', b') \in S_k$.

(ii) $(a, c, b)R(\infty, b')$ whenever $(c, b) \in S_0 - S_1$.

Intuitively, E looks enough like M' for n many steps, and otherwise must look like N; the addition of the point ∞ serves precisely to mimick the possibility of having hit the limit of the model M, and still needing to keep going.

We give this model a valuation $W : (\overline{p}, q) \to \mathcal{P}(E)$ as follows: by saying that $(a, b, c) \in W(i)$ if and only if either $a \in V'(i)$ or $b \in J(i)$, and $(\infty, b) \in W(i)$ whenever $b \in J(i)$.

We now have two things to show:

(i) We have that $(E, (*, b))^q \leftrightarrows (\mathfrak{N}, b)$: indeed, if $*$ is ∞, then whenever bRb', then certainly $(*, b)R(*, b')$; and the converse likewise follows. If instead we consider (a, c, b), and we look at $(a, c, b)R(a', c', b')$, the response is obvious; if bRb' is considered, then either $(c, b) \in S_{k+1}$, and then we can respond by considering a triple (a', c', b') using the bisimulations; or $(c, b) \in S_0 - S_1$, and we respond with (∞, b').

(ii) We have that $(E, (a, c, b)) \leftrightarrows_k (\mathfrak{M}', a)$ where k is the greatest such that $(c, b) \in S_k$: indeed, it is clear that they satisfy the same proposition letters; if $(a, c, b)R(a', c', b')$, then responding with aRa' gives the result by induction. If $(a, c, b) \leftrightarrows_{k+1} a$, and aRa' is considered, then note that $(a', c') \in Z$, and since $(c, b) \in S_{k+1}$, there is some b' such that $(c', b') \in S_k$, i.e., (a', c', b') exists as desired.

Together this implies that:

$$(E, (x', x, y))^q \leftrightarrows (\mathfrak{N}, y) \text{ and } (E, (x', x, y)) \leftrightarrows_n (\mathfrak{M}', x'),$$

which was to show. ■

The construction at play here can best be illustrated in the case where we are dealing with the normal forms of **K** – intransitive trees. In such a case the construction becomes especially transparent.

Example. Suppose that the three models in case are the ones from Figure 1.

We can see that $\mathfrak{M}'$ and $\mathfrak{M}$ are bisimilar by identifying the two branches, and $\mathfrak{M}$ and $\mathfrak{N}$ are 2-bisimilar. Running the construction from the previous lemma, we obtain:

2.4 Uniform definability in other modal logics

The case of **K** reveals that the rigid structure of bisimulation for frames of **K** lends itself to the possibility of uniform definability. The full theory of such "rigidity" is not wholly clear. Nevertheless there are some facts which we can briefly outline.

The case for the modal logics **KB**, **KT** and **KD** follows with some minimal modifications. But the list does not proceed as one might expect: the logics **K4** and **S4** fail to have uniform interpolation. This result, due to Ghilardi and Zawadowski (see e.g. [2, Theorem 6.2]) follows from the combinatorial

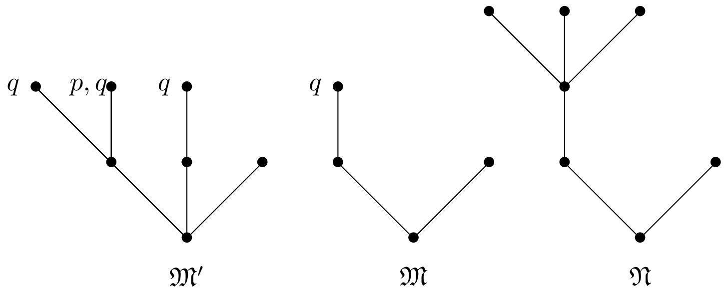

Figure 1: The models $\mathfrak{M}$, $\mathfrak{M}'$ and $\mathfrak{N}$

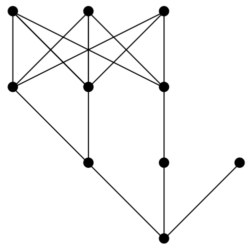

Figure 2: The witness to the combinatorial lemma

conditions outlined before, where it is observed that the equivalent conditions from Theorem 2.13 fail for this logic.

However, the situation is made more complex because for **Grz**, **GL**, **IPC** and other similar logics, the uniform definability property does hold! The reason such a property holds in these cases appears to be that for them, a notion of "rank" can be given to bisimulation, which allows a progress condition within bisimulations, which cannot be obtained in other systems. However, the full theory of uniform definability seems to be far from fully fleshed out.

3 Uniform deductive interpolation

It is of course natural to consider uniform *deductive* interpolation.

Definition 3.1. A logic L has *uniform deductive interpolation* if and only if whenever $\phi(\overline{p},\overline{r}) \vdash_L \psi(\overline{q},\overline{r})$, there two formulas χ_0 and χ_1 in the common language such that:

(i) χ_i are deductive interpolants for $\phi \vdash_L \psi$;

(ii) Whenever μ is a deductive interpolant, then $\chi_0 \vdash \mu$ and $\mu \vdash \chi_1$.

As it can be expected, these notions are interrelated:

Definition 3.2. Let L be a logic. We say that L has a *deduction theorem* if there is a term $t(x)$ such that for each formulas ϕ, ψ, we have

$$\phi \vdash_L \psi \iff \vdash_L t(\phi) \to \psi.$$

Proposition 3.3. *Let L be a logic with a deduction theorem. Then if L has uniform Craig interpolation, then L has uniform deductive interpolation.*

Proof. Exercise. ■

The theory of uniform deductive interpolation is quite extensive, having been developed by Metcalfe and his collaborators (see e.g [5] and [3]).

Unlike the strongly categorical theory of uniform definability, the theory of uniform deductive interpolation has a strongly model theoretic flavour. To explain this we will briefly recall some basic notions of model theory.

3.1 Model Completions

Definition 3.4. Let T be a universal first-order theory. We write $T_\forall$ for the set of universal consequences of T. We say that a theory U is a *cotheory of* T if $U_\forall = T_\forall$. Equivalently, every model of T can be extended to a model of U, and every model of U can be extended to a model of T.

We say that a theory T^* is a *model companion* of T if T and T^* are cotheories and T^* is model-complete.

Our first observation is that model companions, when they exist, are unique:

Proposition 3.5 (Chang and Keisler, Proposition 3.5.13)**.** *For every theory T, it has at most one model companion.*

Proof. Let T^* and T^{**} be two model companions of T. We will show that they have exactly the same set of models. Assume that $\mathfrak{M} \models T^*$. By hypothesis, using the fact that both are cotheories, $\mathfrak{M}$ embeds into $\mathfrak{N}_1 \models T^{**}$; in turn $\mathfrak{N}_1$ embeds into $\mathfrak{N}_2 \models T^*$; hence the composed embedding from $\mathfrak{M}$ to $\mathfrak{N}_2$ is elementary, by model-completeness. Now $\mathfrak{N}_2$ embeds into $\mathfrak{N}_3 \models T^{**}$, and the composed embedding from $\mathfrak{N}_1$ to $\mathfrak{N}_3$ is elementary. Proceeding in this way we obtain a chain, and $\mathfrak{F}$, the colimit of that chain, is a model of T^*, since it is the colimit of an elementary chain of models of T^* ($\mathfrak{N}_{2n}$), and also a model of T^{**}; since $\mathfrak{M}$ embeds elementarily in $\mathfrak{F}$, then $\mathfrak{M} \models T^{**}$. Similarly, every model of T^{**} is a model of T^*. ■

The models of T^* have a concrete model-theoretic description: they are precisely the *existentially closed* models of T.

Proposition 3.6. *Let T be a theory axiomatised by $\forall\exists$ axioms, and T^* its model companion. Then for each model $\mathfrak{M}$, $\mathfrak{M} \models T^*$ if and only if $\mathfrak{M} \models T$ and $\mathfrak{M}$ is existentially closed for T.*

Proof. First assume that $\mathfrak{M} \models T^*$, and $f : \mathfrak{M} \to \mathfrak{N}$ is an embedding where $\mathfrak{N} \models T$; by hypothesis, there is some model $\mathfrak{F}$ such that $g : \mathfrak{N} \to \mathfrak{F}$ is an embedding where $\mathfrak{F} \models T^*$. Note that then $gf : \mathfrak{M} \to \mathfrak{F}$ is an embedding of models of T^*, hence elementary; then this implies that f is an existentially closed embedding. Since $\forall\exists$-axioms are preserved under existential embeddings, $\mathfrak{M} \models T$ as well.

Conversely, assume that $\mathfrak{M}$ is existentially closed for T. We want to show that $\mathfrak{M} \models T^*$. Let $f : \mathfrak{M} \to \mathfrak{N}$ be an embedding where $\mathfrak{N} \models T^*$. Note that since T^* is model complete, then T^* can be axiomatised by $\forall\exists$ axioms; for any such axiom $\forall\overline{x}\exists\overline{y}\phi(\overline{x}, \overline{y})$, if $\overline{a} \in \mathfrak{M}$, then to show that $\mathfrak{M} \models \exists\overline{y}\phi(\overline{a}, \overline{y})$, we can use the existential closure of the embedding to obtain that from the fact that $\mathfrak{N} \models T^*$. ■

The notion of a *model-completion* is a strengthening of the notion model companions; it ensures that, loosely speaking, there is at most one way (up to isomorphism) of embedding the models.

Definition 3.7. Let T be a theory and let T^* be its model companion. We say that T^* is a *model completion* if for every model $\mathfrak{M} \models T$, $T^* \cup Diag(\mathfrak{M})$ is complete.

Definition 3.8. Let T be a theory. We say that T has the *amalgamation property* if whenever $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are three models of T, such that $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ are two embeddings, then there is $\mathfrak{D} \models T$ and embeddings $h_A : \mathfrak{A} \to \mathfrak{D}$ and $h_B : \mathfrak{B} \to \mathfrak{D}$ such that $h_B * g = h_A * f$.

A useful equivalent of the above condition is given Proposition 3.5.18 in Chang and Keisler:

Proposition 3.9. *Let T^* be a model companion of T, a theory axiomatised by $\forall\exists$ axioms. The following are equivalent:*

(i) T^* is a model completion of T;

(ii) T has the amalgamation property.

Proof. First assume that T^* is the model completion of T. Suppose that we are in the situation described by Figure 3

where we assume that all of them satisfy T. Since T^* is a cotheory, there are models $\mathfrak{A}'$ and $\mathfrak{B}'$, extending $\mathfrak{A}$ and $\mathfrak{B}$ respectively; then note that $(\mathfrak{A}', c)_{c\in C}$ and $(\mathfrak{B}', c)_{c\in C}$ are both models of $T^* \cup Diag(C)$, which is a complete theory by assumption. So the models are elementarily equivalent. Note that then we can satisfy the following theory:

$$Diag(\mathfrak{A}', C) \cup Diag(\mathfrak{B}', C) \cup T^*.$$

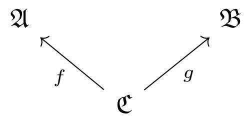

Figure 3: Triangle of Amalgam

Hence let $\mathfrak{C}'$ be a model of that theory. Let $\mathfrak{D}$ be an extension of T, which exists by the latter being cotheories with T^*, and take the composed embeddings; then $\mathfrak{D}$ amalgamates $\mathfrak{A}$ and $\mathfrak{B}$ over C.

Now assume that T has the amalgamation property. Let $\mathfrak{A} \vDash T$, and let $\mathfrak{B}, \mathfrak{C}$ be two models of T^* with embeddings from $\mathfrak{M}$. Then $\mathfrak{A}, \mathfrak{B} \vDash T$, and so we can amalgamate them into $\mathfrak{D}$, which can be extended to a model of T^*; using the composed embeddings we can then show that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent. ■

We also note the following Theorem, which will be of particular use for us, and is Theorem 3.5.20 in Chang and Keisler:

Theorem 3.10. *Let T be a consistent universal theory in a countable language $\mathcal{L}$. Then T has a model completion if and only if condition (i) below holds:*

(i) For every existential formula $\theta(x_1, ..., x_n)$ there is an open formula $\theta'(x_1, ..., x_n)$ such that:

(i) $T \vdash \theta(x_1, ..., x_n) \to \theta'(x_1, ..., x_n)$;

(ii) For every universal formula $\phi(x_1, ..., x_n)$,

$$\text{if } T, \theta' \to \theta \vdash \phi, \text{ then } T \vdash \phi.$$

Moreover if (i) holds, then the theory

$$T' = T \cup \{\forall \overline{x}(\theta' \to \theta) : \theta \text{ is existential } \}$$

is a model completion of T.

3.2 Uniform Interpolation and Algebraic Model Completions

We will need a specific kind of model completion:

Definition 3.11. Let T be a consistent universal theory in a countable language. We say that T has an *algebraic model completion* if in Theorem 7, for each existential formula of the form

$$\theta(\overline{y}) = \exists \overline{x}\ t(\overline{x}, \overline{y}) \approx t'(\overline{x}, \overline{y}).$$

we can choose the quantifier-free formula θ' to be atomic, and whenever

$$\theta(\overline{y}) = \exists \overline{x}\ t(\overline{x}, \overline{y}) \not\approx t'(\overline{x}, \overline{y}).$$

we can choose θ' to be negated atomic.

Having all of this, the following is not very difficult to see:

Proposition 3.12. *Let L be a logic. If L has an algebraic model completion, then L has uniform deductive interpolation.*

Notably, the converse of this is also true:

Proposition 3.13. *Let L be a logic which has uniform deductive interpolation. Then L has an algebraic model completion.*

Proof. See e.g. [2, Theorem 3.11]; we will eventually add a simplification of that proof here, for broad consumption. ■

3.3 Coherence

One key property that has come out of such considerations is the following:

Definition 3.14. Let $\mathcal{K}$ be a variety of algebras. We say that $\mathcal{K}$ is *coherent* if whenever $\mathcal{A}$ is a finitely presented algebra, and $\mathcal{B} \leq \mathcal{A}$ is a finitely generated subalgebra, then $\mathcal{B}$ is finitely presented as well.

Then by the theory, one can for instance derive the following result by Kowalski and Metcalfe [4]:

Theorem 3.15. *Let L be a modal logic such that $\mathsf{Alg}(L)$ is coherent. Then L has a deduction theorem. Moreover, if L has uniform deductive interpolation, then $\mathsf{Alg}(L)$ is coherent.*

As a corollary, **K** does not have uniform deductive interpolation!

Bibliography

[1] Patrick Blackburn, Maarten de Rijke and Yde Venema. *Modal logic.* Vol. Cambridge tracts in theoretical computer science. 53. Cambridge, England: Cambridge University Press, 2002.

[2] Silvio Ghilardi and Marek Zawadowski. *Sheaves, Games, and Model Completions.* Springer Netherlands, 2002. DOI: `10.1007/978-94-015-9936-8`. URL: `https://doi.org/10.1007/978-94-015-9936-8`.

[3] Samuel J. v. Gool and Luca Reggio. 'An open mapping theorem for finitely copresented Esakia spaces'. In: *Topology and its Applications* 240 (May 2018), pp. 69–77. ISSN: 0166-8641. DOI: `10.1016/j.topol.2018.03.006`. URL: `http://dx.doi.org/10.1016/j.topol.2018.03.006`.

[4] Tomasz Kowalski and George Metcalfe. 'Uniform interpolation and coherence'. In: *Annals of Pure and Applied Logic* 170.7 (July 2019), pp. 825–841. ISSN: 0168-0072. DOI: `10.1016/j.apal.2019.02.004`. URL: `http://dx.doi.org/10.1016/j.apal.2019.02.004`.

[5] George Metcalfe, Franco Montagna and Constantine Tsinakis. 'Amalgamation and interpolation in ordered algebras'. In: *Journal of Algebra* 402 (Mar. 2014), pp. 21–82. ISSN: 0021-8693. DOI: `10.1016/j.jalgebra.2013.11.019`. URL: `http://dx.doi.org/10.1016/j.jalgebra.2013.11.019`.

[6] Albert Visser. 'Uniform Interpolation and Layered Bisimulation'. In: *Gödel '96: Logical Foundations of Mathematics, Computer Science and Physics - Kurt Gödel's Legacy.* Ed. by PetrEditor Hájek. Lecture Notes in Logic. Cambridge University Press, 2017, pp. 139–164.