Topic Presentation Ideas

The following are suggestions: students are encouraged to come up with their own topic, if they have a clear idea; in those cases they should discuss with the instructors to check the appropriateness of the topic.

1 Interpolation and Definability Properties

1. **Projective Beth Property**: The Beth property admits some reasonable strengthening which make the notion of implicit and explicit definability better match with the intuitions we might have about it. Discuss the Projective Beth property and relate it to the property of all monomorphisms of a variety being regular. Show that Craig interpolation entails the projective Beth property under suitable assumptions. Give examples of systems without the projective Beth property, and with the projective Beth property but without the Craig interpolation property.

   BONUS: Give a self-contained proof of Maksimova's result that there are 16 extensions of IPC with the projective Beth property.

2. **Lyndon Interpolation and positivity**: A natural strengthening of the Craig interpolation property is the following: given an entailment $\vdash \phi \to \psi$, we can find an interpolant χ such that each ocurrence of a propositional variable p which occurs only positively in ϕ and ψ occurs only positively in χ, and similar for propositional variables ocurring only negatively. Discuss this property and give examples of systems with and without the Lyndon interpolation property.

3. **Uniform Deductive Interpolation**: As we alluded briefly, uniform deductive interpolation is connected to the existence of a *model completion*. Discuss the meaning of model completions, and give examples from the variety of Boolean algebras and the variety of distributive lattices. Then show that **K** does not have a model completion, by connecting this to the property of *coherence*.

   BONUS: Describe explicitly the model completion of one of the following:

   (a) **S5** algebras;

   (b) Implicative semilattices;

   (c) **LC** algebras.

4. **Unary Beth property**: As we have seen in the lectures, there are two versions of the Beth property: the unary one and the infinitary one. We have seen that the infinitary Beth property is equivalent to all epis being onto. Discuss the unary Beth property and relate it to the property of all epimorphisms between finitely presented algebras being onto. Give examples of systems with the unary Beth property but without the infinitary one.

2 Other Algebraic and Logical Properties

1. **Relevance Properties and Rules**: In addition to axioms, one can consider adding *rules* to logical system. The addition of such rules typically leads to rather intricate logical structures, and because of that it is common to study, as a compromise, when certain rules are *admissible*. Briefly discuss the notion of a quasivariety, and the way in which this corresponds a correspondence to rule systems; then discuss notions such as the relevance principles, structural completeness, passive structural completeness, and hereditary structural completeness, relating them to the joint embedding property, and properties of the lattice of subquasivarieties of a rule system.

2. **EDPC and the Deduction Theorem**: We have seen that a key fact establishing the connection between deductive and the Craig interpolation property is the presence of a deduction theorem. Investigate the logical systems that have a deduction theorem, and connect this with EDPC — equationally definable principal congruences. Give a proof that the only varieties of modal algebras with EDPC are the weakly transitive ones.

3. **Local deduction property**: We have seen that the local deduction property, while being mildly weak, plays a crucial roles in the different properties we investigated in this course. Study the local deduction property from an algebraic perspective and if possible, give example of systems that lack this property.

3 Uniform Interpolation

1. **Uniform Interpolation in Intuitionistic logic**: Give a self-contained proof of uniform interpolation for **IPC**.

   BONUS: Extend this proof to the case of the system **KM**.

2. **Uniform Interpolation in transitive modal logics**: Give a self-contained proof that **S4** does not have the uniform interpolation property.

   BONUS: Extend this to show that **S4.1**, or **S4.2**, does not have the uniform interpolation property.

4 Amalgamation in other settings

1. **Amalgamation in non-varieties**: Study amalgamation in finitely generated quasivarieties of Heyting algebras. Give examples or counterexamples of finitely generated quasivarieties of HA with amalgamation.