TOPICS IN ALGEBRAIC LOGIC AND DUALITY THEORY
SEMINAR SHEET 3

- The contents of this seminar sheet will be discussed on **June 10**;
- Pick one or more examples to work out in full detail; you do not need to work on all of the examples.
- Examples are roughly ordered by conceptual difficulty, relative to the material covered in lectures.

(1) **(Craig + Uniform definability = Uniform Craig)** Show that Craig interpolation together with the uniform definability property implies the Uniform Craig property.

(2) **(Characterising formulas in intuitionistic logic)**
Let $(\mathfrak{M}, x)$ and $(\mathfrak{N}, y)$ be two finite intuitionistic models. Show the following: $(\mathfrak{M}, x)$ and $(\mathfrak{N}, y)$ are n-bisimilar if and only if they satisfy the same formulas of implication rank n.
Hint: This will not be so easy. Define a relation $x \leq_{n+1} y$ if whenever $x \leq k$ there is some $y \leq k'$ such that $k \sim_k k'$. Show additionally, by induction, that $x \leq_n y$ if and only if whenever $x \Vdash \phi$ for ϕ a formula of modal depth n, $y \Vdash \psi$. Write $\uparrow^n x = \{y : x \leq_n y\}$. Show that $y \in \uparrow^{n+1} x$ if and only if

$$y \in \bigcap_{v : v \not\leq_n x} \uparrow^n v \to \bigcup_{w : v \not\leq_n w} \uparrow^n w.$$

(3) **(Combinatorial lemma)** Give a proof that the combinatorial lemma holds for the following two cases:
(a) **KD**;
(b) **KB**.

(4) Show that every Boolean algebra embeds into an existentially closed Boolean algebra.
Hint: If you know model theory, prove this directly. If you do not, prove the following easier statement: every finite Boolean algebra embeds into an atomless Boolean algebra.