Topics in Algebraic Logic and Duality Theory
Lecture 1

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May 29, 2025

Plan for the Day

- Announcements.
- Recap of Heyting algebras (today's focus).
- Free algebras.
- Deductive and Craig interpolation.
- Amalgamation and Super-Amalgamation.

Announcements

1. On Wednesday there will be the first seminar.

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2. A seminar instruction sheet – with suggested topics and reading – has been posted.

Modal and Intuitionistic Logics

We start with intuitionistic and modal logic:

Definition

Let $\overline{p} = \{p_1, ..., p_n\}$ be some finite set of propositional letters, and M some language (either modal or intuitionistic). We denote by $\mathcal{L}_M(\overline{p})$ the set of M formulas over $\overline{p}$.

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Definition

Let L be a set of formulas in M. We say that L is a logic if:

1. Whenever M is the intuitionistic language, $L \supseteq \mathsf{IPC}$ and L is closed under uniform substitution and Modus Ponens;
2. Whenever M is the modal language, $L \supseteq \mathsf{K}$ and L is closed under uniform substitution, necessitation and Modus Ponens.

Recap

It will be convenient for future purposes to consider relations between formulas:

Definition

Let M be a language, and L be a logic, and let Δ be a set of formulas. We say that a sequence $(\phi_0, ..., \phi_n)$ is an L-derivation with hypotheses in Δ if:

1. Whenever M is intuitionistic, either ϕ_i is a substitution instance of an axiom of L, or $\phi_i \in \Delta$, or ϕ_i is obtained from ϕ_j, ϕ_k for $j, k < i$ by applying Modus Ponens;
2. Whenever M is modal logic, the above with the additional possibility that ϕ_i is obtained from ϕ_k for $k < i$ by applying necessitation.

Given $\Delta \cup \{\phi\}$ a set of M-formulas we write:

$$\Delta \vdash_L \phi$$

to mean that there is some L-derivation of ϕ with hypotheses in Δ.

Example

In modal logic $\mathbf{K}$ we have:

$$\phi \vdash_{\mathbf{K}} \Box\phi.$$

However, $\nvdash_{\mathbf{K}} p \to \Box p$. So there are L-derivations which are not reducible to theorems!

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Example

In IPC we have for each pair of formulas ϕ, ψ:

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Derivations and deduction-detachment

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Example

In modal logic $\mathbf{S4}$ we have:

$$\phi \vdash_{\mathbf{S4}} \psi \iff \vdash_{\mathbf{S4}} \Box\phi \to \psi$$

Algebras

Algebraic semantics

Given M a language, we will focus on the kinds of algebras which will be useful for the system at hand:

1. If M is intuitionistic logic, we will focus on Heyting algebras and its subvarieties (like Boolean algebras);
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Recall that an algebra A is called *subdirectly irreducible* if $\mathsf{Con}(A)$ has a second least element; equivalently Δ is completely meet irreducible. It is called *finitely subdirectly irreducible* if Δ is meet irreducible. For Heyting algebras these are the duals of *strongly rooted* and *rooted* Esakia spaces.

Given two (Heyting or modal) algebras A, B of the same type, and a surjective homomorphism $f : A \to B$ (a 'quotient'), there is a way of representing this 'internally' in A: with *filters*. Importantly we recall a notion that was not heavily stressed in the lectures in MSL:

Definition

Let $(A, \Box)$ be a modal algebra. We say that a filter $F \subseteq A$ is a *modal filter* if whenever $a \in F$ then $\Box a \in F$.

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Theorem

There are 1-1 correspondences between the following:

1. *Filters on a Heyting algebra A and quotients $f : A \to B$;*
2. *Modal filters on a modal algebra A and quotients $f : A \to B$.*

Given a quotient $f : A \to B$ we can form a filter

$$F_f := f^{-1}[1],$$

which, if A, B are modal algebras and f is a modal homomorphism, will be a modal filter.

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Conversely, if $F \subseteq A$ is a filter, we can define a quotient algebra A/F by the following equivalence:

$$a \sim_F b \iff a \leftrightarrow b \in F.$$

You should check this gives a quotient in the appropriate cases.

When working with duality, filters admit a nice representation:

Theorem

There is a dual correspondence between the following:

1. *Filters on a Heyting algebra H, and closed upsets in X_H;*
2. *Modal filters on a modal algebra A, and closed generated subframes of (X_A, R).*

Distributive lattices *do not follow the previous pattern*. Their quotients are not given by filters:

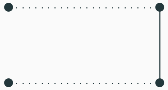

Figure 1: Example of distributive lattice quotient that is not a filter

Free algebras

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Definition

Let X be a set of propositional letters and L be a logic. We denote by $\mathcal{F}_L(X)$ the algebra of formulas on X modulo L provability.

In other words, elements of this algebra are equivalence classes of formulas $[\phi]_L$, where:

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These algebras enjoy a special categorical property:

Lemma

Let L be a logic, and $A \in \mathsf{Alg}(L)$ and $v : X \to A$ be any map. Then there is a unique homomorphism $\overline{v} : \mathcal{F}_L(X) \to A$ such that $\overline{v}(x) = v(x)$ for each $x \in X$.

Free algebras

Free algebras have the advantage that they allow us to reason syntactically. Every algebra can be thought of as a quotient of a free algebra:

Lemma

Let L be a logic, and $A \in \mathsf{Alg}(L)$. Then there is some X and a surjective homomorphism $\mathcal{F}_L(X) \to A$.

Proof.

Let $X = A$; then use the previous lemma. $\square$

The logical import of free algebras is that they allow us to reason about logic from the point of view of a single algebraic model:

Lemma

Let L be a logic in a language M, and $\phi \in \mathcal{L}_M(\overline{p})$.Then we have that $\phi \in L$ if and only if $[\phi]_L = 1$ in $\mathcal{F}_L(\overline{p})$.

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Consequently, we have:

$$[\phi]_L \leqslant [\psi]_L \iff \phi \to \psi \in L.$$

Now let us consider some *Interpolation Properties*:

Definition

Let M be a language, and L be a logic. We say that L has the:

1. *Craig Interpolation Property* if and only if for each pair of formulas $\phi \in \mathcal{L}_M(\overline{p}, \overline{r})$ and $\psi \in \mathcal{L}_M(\overline{q}, \overline{r})$, if $\vdash_L \phi \to \psi$ then there is a formula $\chi \in \mathcal{L}_M(\overline{r})$ such that:
$$\vdash_L \phi \to \chi \text{ and } \vdash_L \chi \to \psi.$$
2. *Deductive interpolation property* if and only if for each pair of formulas $\phi \in \mathcal{L}_M(\overline{p}, \overline{r})$ and $\psi \in \mathcal{L}_M(\overline{q}, \overline{r})$, if $\phi \vdash_L \psi$ then there is a formula $\chi \in \mathcal{L}_M(\overline{r})$ such that:
$$\phi \vdash_L \chi \text{ and } \chi \vdash_L \psi.$$

Theorem

The logic CPC *has Craig (deductive) interpolation.*

Proof.

Assume that $\vdash_{\mathsf{CPC}} \phi \to \psi$. Consider:

$$\chi := \phi(\top, \overline{r}) \vee \phi(\bot, \overline{r}).$$

Then by induction on the structure of formulas, using negation normal form, we can show that $\phi \leqslant \chi$; and by uniform substitution, since $\vdash \phi \to \psi$, then $\vdash \chi \to \psi$. This shows the result. $\square$

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This proof proves both Craig and deductive interpolation; in fact it shows something much stronger which we will see again later.

Amalgamation

Definition

Let $\mathcal{K}$ be a class of algebras. We say $\mathcal{K}$ has the *amalgamation property* if whenever (A, B_1, B_2, f_1, f_2) is a tuple of algebras in $\mathcal{K}$, where f_1, f_2 are injective (an amalgam), there is some algebra $C \in \mathcal{K}$ and a pair of injective morphisms making the diagram commute.

$$
\begin{tikzcd}
& C & \\
B_1 \arrow[ur, hook, dashed] & & B_2 \arrow[ul, hook', dashed] \\
& A_0 \arrow[ul, hook', "f_1"] \arrow[ur, hook, "f_2"'] &
\end{tikzcd}
$$

Figure 2: Amalgamation Diagram

Let us give a simple example of this, using duality.

Theorem

The class of finite Boolean algebras has the amalgamation property.

Proof.

Let (A, B_1, B_2, f_1, f_2) be an amalgam of finite Boolean algebras. By duality, we obtain a tuple (X, Y_1, Y_2, g_1, g_2) where g_1 and g_2 are surjective functions.

Amalgamation for finite Boolean algebras

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$$Eq(g_1, g_2) = \{(x, y) \in X_1 \times X_2 : g_1(x) = g_2(y)\};$$

this is a set, and its projection on each coordinate is surjective: if $x \in X_1$, then $g_1(x) = g_2(y)$ by surjectivity of g_2, so $(x, y) \in Eq(g_1, g_2)$; by duality, this ensures amalgamation. $\square$

This property can be made stronger:

Definition

A class $\mathcal{K}$ of algebras has the super-amalgamation property if whenever (A, B_1, B_2, f_1, f_2) is an amalgam, there is an algebra C and a pair of injective morphisms p_1, p_2 witnessing the amalgamation property, and satisfying the following additional property: for each $a \in A_1$ and $b \in A_2$ whenever $p_1(a) \leqslant p_2(b)$ then there is some $c \in A_0$ such that $a \leqslant f_1(c)$ and $f_2(c) \leqslant b$.

Interpolation and Amalgamation

Craig interpolation and super-amalgamation

We will now relate the properties we have just introduced. First we note that given $\overline{p}, \overline{q}, \overline{r}$ three sets of propositional variables, we have that the following diagram always commutes:

$$\begin{tikzcd}
 & \overline{F}_L(\overline{p}, \overline{q}, \overline{r}) & \\
\mathcal{F}_L(\overline{p}, \overline{r}) \arrow[ur, hook] & & \mathcal{F}_L(\overline{q}, \overline{r}) \arrow[ul, hook'] \\
 & \mathcal{F}_L(\overline{r}) \arrow[ul, hook', "i"] \arrow[ur, hook, "i"'] &
\end{tikzcd}$$

Figure 3: Amalgamation of free algebras

Then we have:

Theorem

The diagram above is a super-amalgamation if and only if L has the Craig interpolation property.

Maksimova's Characterization of Interpolation

Fix now a language M.

Theorem

The following are equivalent for L a logic:

1. *L has the deductive interpolation property;*
2. *$\mathbf{V}_L$ has the amalgamation property.*

Proof.

(2) implies (1): for this we will need to use a lemma:

Lemma

Assume that $\phi(\overline{p}, \overline{r}) \vdash_L \psi(\overline{q}, \overline{r})$, but there is no $\chi(\overline{r})$ which interpolates them deductively. Then there is a pair of filters $F_1 \subseteq \mathcal{F}_L(\overline{p}, \overline{r})$ and $F_2 \subseteq \mathcal{F}_L(\overline{q}, \overline{r})$, such that $F_0 = F_1 \cap \mathcal{F}_L(\overline{r}) = F_2 \cap \mathcal{F}_L(\overline{r})$, $[\phi]_L \in F_1$ and $[\psi]_L \notin F_2$.

Assume then that $\phi(\overline{p}, \overline{r}) \vdash_L \psi(\overline{q}, \overline{r})$ has no interpolant. By the Lemma, let F_0, F_1, F_2 be given. $\square$

Proof.

Now consider the following diagram:

$$\begin{tikzcd}
& D & \\
\mathcal{F}_L(\overline{p},\overline{r})/F_1 \arrow[ur, dashed] & & \mathcal{F}_L(\overline{q},\overline{r})/F_2 \arrow[ul, dashed] \\
& \mathcal{F}_L(\overline{r})/F_0 \arrow[ul, "i_1"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Figure 4:

The conditions on the filters F_1, F_2 ensure that i_1, i_2 are injective. Hence there is an algebra D, and maps h_1, h_2. Now consider the valuation $v : \overline{p,q,r} \to D$ given by the diagram,
i.e.$v(\overline{p}) = h_1(\overline{p}),\ v(\overline{q}) = h_2(\overline{q})$ and $v(\overline{r}) = h_1(\overline{r}) = h_2(\overline{r})$.
Then we have that for each formula $\phi(\overline{p}, \overline{r})$, $v(\phi) = h_1(\phi)$, and similar for h_2.

Maksimova's Characterization of Interpolation (cont.d)

Proof.

Now consider the following diagram:

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Then we have that for each formula $\phi(\overline{p},\overline{r})$, $v(\phi) = h_1(\phi)$, and similar for h_2.
Now since $\phi \in F_1$, $v(\phi) = 1$; hence by hypothesis, $v(\psi) = 1$; so $h_2(\psi) = 1$, so by assumption, $\psi \in F_2$ – a contradiction. $\square$

Proof.

(1) implies (2): Now assume that L has deductive interpolation. Let (A, B_1, B_2, f_1, f_2) be an amalgam. Fix some presentation of these as quotients of free algebras.

$$
\begin{tikzcd}
 & & \mathcal{F}_L(B_1 \cup B_2) \arrow[d, dashed, two heads] & & \\
\mathcal{F}_L(B_1) \arrow[urr, hook] \arrow[d] & & C & & \mathcal{F}_L(B_2) \arrow[ull, hook'] \arrow[d, two heads] \\
B_1 \arrow[urr, dashed, hook] & & \mathcal{F}_L(A) \arrow[ull, hook', "k_1"] \arrow[urr, hook, "k_2"'] \arrow[d, two heads] & & B_2 \arrow[ull, dashed] \\
 & & A \arrow[ull, hook'] \arrow[urr, hook] & &
\end{tikzcd}
$$

Figure 5: Interpolation diagram

In particular, fix F_1 the filter producing B_1, F_2 the filter producing B_2, and F_0 the filter producing A. $\square$

Proof.

C is obtained from $G = \mathsf{Fil}(F_1 \cup F_2)$ and $k_i : B_i \to C$ is obtained by sending $b \in B_i$ to itself in $\mathcal{F}_L(B_1 \cup B_2)$, and composing that with the quotient to C; this is well-defined because $F_1 \subseteq G$.

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The diagram commutes – this is a matter of diagram chasing. The key is to show that k_1 and k_2 are injective.

For that assume that $k_1(b) = 1$; we need to show that $b = 1$. The former means that $b \in G$, so there is some $c \in F_1$ and $d \in F_2$ such that $c \wedge d \leqslant b$. Hence $\vdash_L d \to (c \to b)$, so it follows that $d \vdash_L c \to b$; then by deductive interpolation there is some k in the language of A such that $d \vdash_L k$ and $k \vdash_L c \to b$. Since $d \in F_2$, then $k \in F_2$; since the inclusion of A into B_2 is injective, then $k \in F_0$, so $[k] = 1$ in B_1. Thus $[c \to b] = 1$ in B_1, and since $c \in F_1$, then $b \in F_1$ as well. □

Theorem

The following are equivalent for a logic L:

1. *L has the Craig interpolation property.*
2. $\mathsf{Alg}(L)$ *has the superamalgamation property.*

This equivalence can be proven using a similar (but simpler) strategy.

Equivalences in superintuitionistic logics

In the case of intuitionistic logic we can obtain a stronger result:

Theorem

The following are equivalent for a superintuitionistic logic L:

1. *L has the Craig interpolation property;*
2. *L has amalgamation;*
3. *Any triple (H, H_0, H_1) where H, H_0, H_1 are finitely subdirectly irreducible is amalgamable in L.*

To prove the equivalence with the latter, we can use the following lemma:

Lemma

Let $A_0 \leqslant A_1, A_2$ be M-algebras. Suppose that $a \in A_1$ and $b \in A_2$ but there exists no $c \in A_0$ such that both $a \leqslant c$ and $c \leqslant b$. Then there exist prime filters $F_1 \subseteq A_1$ and $F_2 \subseteq A_2$ such that $a \in F_1$, $b \notin F_2$ and $F_1 \cap A_0 = F_2 \cap A_0$.

Examples and Counterexamples

We will show:

Theorem

IPC *has the Craig interpolation property.*

Proof.

By Maksimova's characterization, it suffices to show amalgamation. We use duality.

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Let (X, Y_0, Y_1, g_1, g_2) be a diagram of Esakia spaces where g_1, g_2 are p-morphisms. Consider:

$$Eq(g_1, g_2) = \{(x, y) \in Y_1 \times Y_2 : g_1(x) = g_2(y)\},$$

understood as a poset with the coordinatewise order.

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understood as a poset with the coordinatewise order. Note that $\pi_{Y_1} : Eq(g_1, g_2) \to Y_1$ is a p-morphism: if $\pi_{Y_1}(x, y) \leqslant z$, then $g_1(x) \leqslant g_1(z)$; then $g_2(y) \leqslant g_1(z)$, so since g_2 is a p-morphism, there is some $y \leqslant w$ such that $g_1(z) = g_2(w)$. Then $(x, y) \leqslant (z, w)$, and $\pi_{Y_1}(z, w) = z$.

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Moreover, note that with π_{Y_1} and π_{Y_2}, the diagram commutes. □

Proof.

Now consider $\mathsf{Up}(Eq(g_1, g_2))$; this is a Heyting algebra. Moreover, because π_{Y_1} is surjective a p-morphism, $\pi_{Y_1}^{-1} : \mathsf{ClopUp}(Y_1) \to \mathsf{Up}(Eq(g_1, g_2))$ is an injective Heyting algebra homomorphism.

Proof.

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Moreover the diagram commutes. So this provides an amalgam as desired.

$\square$

Craig interpolation for other systems

The above proof works in a very similar way to prove Craig interpolation for $\mathbf{K}$.

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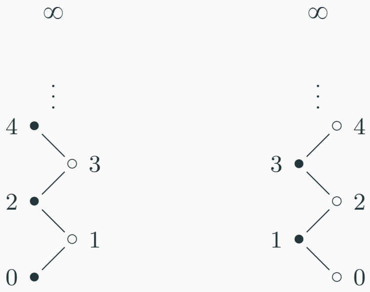
For extensions, it often needs tweaking: we will see this in the seminar on Wednesday.

Let us give an example of failure of deductive interpolation for $\mathbf{S4.3}$, again using failure of amalgamation. Let $\bullet$ denote irreflexive points and $\circ$ denote reflexive points.



Y_1 Y_2

Let X be a two element cluster, $\{E, O\}$. Define p-morphisms $g_i : Y_i \to X$ by sending all the evens to E and all the odds to O.

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Suppose that Z amalgamates this diagram. Let $p_i : Z \to Y_i$ be the maps. First note that if $k \in Z$, and $p_i(k)$ is a natural number, then k is irreflexive.

Let X be a two element cluster, $\{E, O\}$. Define p-morphisms $g_i : Y_i \to X$ by sending all the evens to E and all the odds to O.

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Now let $p_1(k_0) = 0$ and let k_2 be minimal above k_0 such that $p_1(k_2) = 2$. By the p-morphism condition let $p_1(k_1) = 1$, and successively $p_1(k_1') = 1$; note that by linearity, and order preservation, we must have $k_0 R k_1 R k_1' R k_2$.

Failure of Interpolation for S4.3

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Now since k_0 and k_2 map to evens, and k_1 maps to an odd, by the diagram commuting, we must have that $p_2(k_0) \neq p_2(k_2)$, say $p_2(k_0) = 2n$. By the p-morphism condition there is some $k_1 R z_1 R z_2$, such that $p_2(z_1) = 2n + 1$ and $p_2(z_2) = 2n + 2$. Thus $p_1(z_2) \neq 0$, and so by construction, $k_2 R z$. But then $p_2(k_2) R p_2(z_2)$, which means by the arguments given that $p_2(k_2) = 2n + 2$.

Failure of Interpolation of S4.3 (cont.d)

Then we have that $p_2(k_0) = 2n$ and $p_2(k_2) = 2n + 2$. But we have that then $p_2(k_1) = p_2(k_1')$. By order preservation, then $p_2(k_1) R p_2(k_1')$, which contradicts the fact that this point is irreflexive. $\square$

Next time

- Beth's definability property.

Next time

- Beth's definability property.
- Epimorphism surjectivity.

Next time

- Beth's definability property.
- Epimorphism surjectivity.
- (?)

Thank you!
Questions?