Topics in Algebraic Logic and Duality Theory
Lecture 2

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June 5, 2025

Plan for the Day

- Beth definability property.
- Epimorphism surjectivity.

Beth definability

Definition

Let L be a logic.

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$$\Gamma(\overline{p}, r_1), \Gamma(\overline{p}, r_2) \vdash_L r_1 \leftrightarrow r_2.$$

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The logic L has the *Beth property* if for any set of formulas $\Gamma(\overline{p}, r)$, if Γ implicitly defines r, then there is an explicit definition of r relative to Γ.

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In CPC, consider $\Gamma(p_1, p_2, r) = \{r \to p_1, r \to p_2, p_1 \to (p_2 \to r)\}$.

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Γ implicitly defines r. $\phi_r = p_1 \wedge p_2$ is an explicit definition relative to Γ.
However, in the implicative fragment of CPC, such an explicit is not possible.

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Theorem (Kreisel, 1960)

Any intermediate logic L has the Beth property.

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Assume that Γ implicitly define r, that is,

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Substituting r for r_1 and $\top$ for r_2, we obtain $\Gamma(\overline{p}, r), \Gamma(\overline{p}, \top) \vdash_L r$.

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$$\Gamma(\overline{p}, r) \vdash_L \Gamma(\overline{p}, \top) \to r.$$

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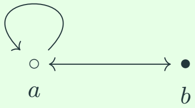
Example

Let L be the logic of the frame below. Note that it is weakly transitive.



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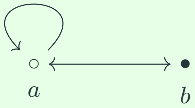

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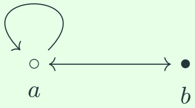

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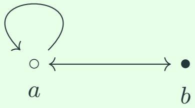

Consider $\Gamma(r) = r \leftrightarrow \Box\neg r$. The only valuation validating it is $r \mapsto \{b\}$, thus Γ implicitly defines r.

If r were explicitly definable, it would be by a variable free formula. However, every variable free formula is equivalent to $\top$ or $\bot$ (by induction).

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Proposition

If a logic L has the infinitary Beth property, then it has the Beth property.

Local deduction property

Definition

A logic L has the *local deduction property* if for each pair of formulas ϕ, ψ, there is a formula δ_ϕ in the language of ϕ such that for each formula ξ,

1. $\xi, \phi \vdash_L \psi$ iff $\xi \vdash_L \delta_\phi \to \psi$,
2. $\phi, \delta_\phi \to \xi \vdash_L \xi$.

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Proposition (Exercise)

Let L be a logic with the local deduction property. If L has the Craig interpolation property, then L has the deductive interpolation property.

Craig and Beth, hand in hand

Theorem

Let L be a (compact, conjunctive) logic with the local deduction property. If L has the Craig interpolation property, then L has the (infinitary) Beth property.

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Let L be a (compact, conjunctive) logic with the local deduction property. If L has the Craig interpolation property, then L has the (infinitary) Beth property.

Proof.

See seminar 2. $\square$

Epimorphism surjectivity

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Definition

A map $h \colon A \to B$ is *epic* (or an *epimorphism* if for all maps $f_1, f_2 \colon B \to C$, we have $f_1 \circ h = f_2 \circ h$ implies $f_1 = f_2$.

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In general, onto maps are epic. A variety of algebras is said to have the *epimorphism surjectivity property* if the converse is true (for its morphisms).

Example

The varieties BA, HA, MSL and Lat have epimorphism surjectivity.
The variety DL doesn't, as is witnessed by the map $\mathbf{3} \to \mathbf{2} \times \mathbf{2}$.

Theorem (Hoogland, 2000)

Let L be a logic and K the variety of its algebras. Then L has the Beth property iff K has epimorphism surjectivity.

Beth definability and epimorphism surjectivity

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First assume that K has epimorphism surjectivity, and let $\Gamma(\overline{p}, \overline{r})$ such that for every corresponding pair $r_1 \in \overline{r_1}$, $r_2 \in \overline{r_2}$,

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$$\begin{tikzcd}
F(\overline{p}) \arrow[d, hook, "\iota"'] \arrow[dr, "\pi \circ \iota"] & \\
F(\overline{p}, \overline{r}) \arrow[r, two heads, "\pi"] & F(\overline{p}, \overline{r})/\Gamma
\end{tikzcd}$$

The map $\pi \circ \iota$ is epic.

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The map $\pi \circ \iota$ is epic. Take $h_1, h_2 \colon F(\overline{p}, r)/\Gamma \to A$ such that $h_1 \circ \pi \circ \iota = h_2 \circ \pi \circ \iota$. Clearly h_1 and h_2 agree on $\overline{p}$, and it is sufficient to show $h_1(r) = h_2(r)$ for each $r \in \overline{r}$. $\square$

Beth definability and epimorphism surjectivity (cont.d)

Proof.

We define a valuation

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v\colon F(\overline{p},\overline{r_1},\overline{r_2}) &\to A \\
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Observe that for $\gamma \in \Gamma$,

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Therefore v sends $\Gamma(\overline{p}, \overline{r_1})$ to 1.

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Observe that for $\gamma \in \Gamma$,

$$v(\gamma(\overline{p},\overline{r_1})) = \gamma(v(\overline{p}), v(\overline{r_1})) = \gamma(h_1(\overline{p}), h_1(\overline{r})) = h_1(\gamma(\overline{p},\overline{r})) = h_1(1) = 1.$$

Therefore v sends $\Gamma(\overline{p},\overline{r_1})$ to 1. Similarly for $\Gamma(\overline{p},\overline{r_2})$.

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Beth definability and epimorphism surjectivity (cont.d)

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This proves that $h_1 = h_2$, thus $\pi \circ \iota$ is epic. $\square$

Beth definability and epimorphism surjectivity (cont.d)

Proof.

As $\pi \circ \iota$ is epic, it is surjective.

Beth definability and epimorphism surjectivity (cont.d)

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As $\pi \circ \iota$ is epic, it is surjective. Thus there is $\phi_r \in F(\overline{p})$ such that $\pi \circ \iota(\phi_r) = r$ for all $r \in \overline{r}$. We have $\phi_r = r$ in $F(\overline{p}, \overline{r})/\Gamma$ iff $\Gamma(\overline{p}, \overline{r}) \vdash_L r \leftrightarrow \phi_r$.

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A \arrow[d, two heads, "\pi"'] \arrow[dr, "h"] & \\
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h is epic iff ι is, and h is onto iff ι is.

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Beth definability and epimorphism surjectivity (cont.d)

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Let A be a subalgebra of B such that the inclusion is epic.

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This prove that Γ implicitly defines $\overline{r}$. $\square$

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As Γ implicitly define $\overline{r}$, it must also explicitly define it.

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Thank you!
Questions?