Topics in Algebraic Logic and Duality Theory
Lecture 3

Rodrigo N. Almeida, Simon Lemal

June 11, 2025

Plan for the Day

- Announcements.
- Local Uniform interpolation and Adjoints.
- Deductive Uniform interpolation and Model Completions.

Announcements

1. This week you should start thinking about a topic.

Announcements

1. This week you should start thinking about a topic.
2. Topics should be decided by the middle of next week, to ensure you have enough time.

Uniform interpolation

Interpolation seen uniformly

Last week we saw a proof of interpolation for CPC which proved something stronger: that given a formula $\phi(\overline{p}, q)$, we can find a formula $\chi(\overline{p})$ which *always works as an interpolant.*

Last week we saw a proof of interpolation for CPC which proved something stronger: that given a formula $\phi(\overline{p}, q)$, we can find a formula $\chi(\overline{p})$ which *always works as an interpolant*.

As we will see, this situation is not exclusive of classical logic.

Local Uniform Interpolation

Adjoints

Let $\mathcal{K}$ be a class of ordered algebras, possessing free algebras; we will say that $\mathcal{K}$ has the *uniform definability property* if whenever $[k] = \{p_1, ..., p_k\}$ then the inclusion:

$$i : \mathcal{F}_{\mathcal{K}}([k]) \hookrightarrow \mathcal{F}_{\mathcal{K}}([k+1]),$$

has a left and a right adjoint;

Adjoints

Let $\mathcal{K}$ be a class of ordered algebras, possessing free algebras; we will say that $\mathcal{K}$ has the *uniform definability property* if whenever $[k] = \{p_1, ..., p_k\}$ then the inclusion:

$$i : \mathcal{F}_{\mathcal{K}}([k]) \hookrightarrow \mathcal{F}_{\mathcal{K}}([k+1]),$$

has a left and a right adjoint;

Explicitly, this means that there are two order preserving maps $\forall_i : \mathcal{F}_{\mathcal{K}}([k+1]) \to \mathcal{F}_{\mathcal{K}}([k])$ and $\exists_i : \mathcal{F}_{\mathcal{K}}([k+1]) \to \mathcal{F}_{\mathcal{K}}([k])$, such that for each $a \in \mathcal{F}_{\mathcal{K}}([k])$ and $b \in \mathcal{F}_{\mathcal{K}}([k+1])$ we have:

$$\exists_i(b) \leqslant a \iff b \leqslant i(a) \text{ and } i(a) \leqslant b \iff a \leqslant \forall_i(b).$$

Definition

We say that a logic L has the *uniform Craig interpolation property* if and only if given $\phi(\overline{p}, \overline{r})$, there are two formulas χ_L and χ_R in the language of $\overline{p}$, such that for all $\psi(\overline{p}, \overline{r})$, if $\vdash \phi \to \psi$ then χ_L is a (left) uniform interpolant for this sequent; and whenever $\vdash \psi \to \phi$ then χ_R is a (right) uniform interpolant for this sequent.

Proposition

Let L be a logic with the Craig interpolation property and the uniform definability property. Then L has the uniform Craig interpolation property.

(Exercise)

Proposition

Let L be a logic with the Craig interpolation property and the uniform definability property. Then L has the uniform Craig interpolation property.

(Exercise)

Proposition

Let L be a locally tabular logic. Then L has the uniform definability property.

Proposition

Let L be a logic with the Craig interpolation property and the uniform definability property. Then L has the uniform Craig interpolation property.

(Exercise)

Proposition

Let L be a locally tabular logic. Then L has the uniform definability property.

What to do for non-locally tabular logics?

N-bisimulations

Definition

Let $\phi \in \mathcal{L}$. We define the *modal depth* of ϕ by recursion as follows:

1. $md(p) = 0$;
2. $md(\phi \wedge \psi) = \max(md(\phi), md(\psi))$ and $md(\neg\phi) = md(\phi)$;
3. $md(\Box\phi) = md(\phi) + 1$.

Definition

Let $\phi \in \mathcal{L}$. We define the *modal depth* of ϕ by recursion as follows:

1. $md(p) = 0$;
2. $md(\phi \wedge \psi) = \max(md(\phi), md(\psi))$ and $md(\neg\phi) = md(\phi)$;
3. $md(\Box\phi) = md(\phi) + 1$.

Definition

Let $(\mathfrak{M}, x) = (W, R, V)$ and $(\mathfrak{N}, y) = (W', R', V')$ be two models over $\overline{p}$. We say that a relation $S_n \subseteq W \times W'$ is an *n-bisimulation* based on (x, y) if there are relations $S_n \subseteq ... \subseteq S_k \subseteq ... \subseteq S_0$ for each $0 \leqslant k \leqslant n$, such that xS_ny and:

1. Whenever wS_0w' then $w \in V(p) \iff w' \in V'(p)$ for each $p \in \overline{p}$;
2. For each $k < n$, whenever $wS_{k+1}w'$ and wRv there is some $w'Rv'$ such that vS_kv'.
3. For each $k < n$, whenever $wS_{k+1}w'$ and $w'Rv'$ there is some wRv such that vS_kv'.

One of the reasons we care about this is the following:

Proposition

Given two models $\mathfrak{M}, x$, and $\mathfrak{N}, y$ we have that $\mathfrak{M}, x \leftrightarrows_n \mathfrak{N}, y$ if and only if $\mathfrak{M}, x$ and $\mathfrak{N}, y$ satisfy the same formulas of modal depth n (resp. implication rank n).

Proof.

(See board) $\square$

N-bisimulation and modal equivalence

One of the reasons we care about this is the following:

Proposition

Given two models $\mathfrak{M}, x$, and $\mathfrak{N}, y$ we have that $\mathfrak{M}, x \leftrightarrows_n \mathfrak{N}, y$ if and only if $\mathfrak{M}, x$ and $\mathfrak{N}, y$ satisfy the same formulas of modal depth n (resp. implication rank n).

Proof.

(See board) $\square$

From this we derive:

Proposition

Let L be a logic with the finite model property. Let $\mathcal{K}$ be a class of (finite or infinite) models of L, over $\overline{p}$. Then the following are equivalent:

1. *$\mathcal{K}$ is closed under n-bisimulation;*
2. *There is a formula ϕ of modal depth at most n such that for each $\mathfrak{M}, x$ a finite model, $\mathfrak{M}, x \Vdash \phi$ if and only if $(\mathfrak{M}, x) \in \mathcal{K}$.*

One way to look at uniform definability is through *bisimulation quantifiers*:

$$\mathfrak{M}, x \Vdash \tilde{\exists} p_i\, \phi \iff \text{there is a model } (\mathfrak{N}, y), \mathfrak{M}^{p_i}, x \leftrightarroweq \mathfrak{N}, y, \text{ and } \mathfrak{N}, y \Vdash \phi.$$

One way to look at uniform definability is through *bisimulation quantifiers*:

$$\mathfrak{M}, x \Vdash \tilde{\exists} p_i\, \phi \iff \text{there is a model } (\mathfrak{N}, y), \mathfrak{M}^{p_i}, x \leftrightarrows \mathfrak{N}, y, \text{ and } \mathfrak{N}, y \Vdash \phi.$$

Proposition

For each model $(\mathfrak{M}, x)$ over $(\overline{p}, q)$, and each formula $\phi(\overline{p}, q)$ we have:

1. $\mathfrak{M}, x \Vdash \phi(\overline{p}, q) \to \tilde{\exists} q\, \phi(\overline{p}, q)$;
2. *Whenever* $\mathfrak{M}, x \Vdash \tilde{\exists} q\, \phi(\overline{p}, q) \to \chi$, *then* $\mathfrak{M}, x \Vdash \phi(\overline{p}, q) \to \chi(\overline{p})$.

Moreover, if for each finite model, $\mathfrak{M}, x \Vdash \phi \to \psi$, *then for each finite model* $\mathfrak{M}, x \Vdash \tilde{\exists} q\, \phi \to \tilde{\exists} q\, \psi$.

Bisimulation quantifier elimination

Corollary

Assume that L is a logic which has the FMP. Suppose that for each $\phi(\overline{p}, q)$ in the language $\mathcal{L}(M)(\overline{p}, q)$, there is a formula $\psi(\overline{q})$ in the same language such that for each finite model $\mathfrak{M}, x$:

$$\mathfrak{M}, x \Vdash \psi(\overline{p}) \iff \mathfrak{M}, x \Vdash \tilde{\exists} q\ \phi(\overline{p}, q).$$

Then L has the uniform definability property.

Corollary

Assume that L is a logic which has the FMP. Suppose that for each $\phi(\overline{p}, q)$ in the language $\mathcal{L}(M)(\overline{p}, q)$, there is a formula $\psi(\overline{q})$ in the same language such that for each finite model $\mathfrak{M}, x$:

$$\mathfrak{M}, x \Vdash \psi(\overline{p}) \iff \mathfrak{M}, x \Vdash \tilde{\exists} q\ \phi(\overline{p}, q).$$

Then L has the uniform definability property.

We note that under very reasonable assumptions, the above is an equivalence.

Distilling combinatorial conditions

Theorem

Let L be a logic with the FMP. Then L has uniform definability whenever the following holds:

1. *Whenever $(\mathfrak{M}, x) \in \pi_q[\mathcal{K}]$, so that $(\mathfrak{M}, x) \leftrightarrows (\mathfrak{M}', x')^q$ and $(\mathfrak{M}, x) \leftrightarrows_k (\mathfrak{N}, y)$, then there is some model $(\mathfrak{N}', y')$ over $(\overline{p}, q)$ such that $(\mathfrak{N}', y')^q \leftrightarrows (\mathfrak{N}, y)$ and $(\mathfrak{N}', y') \leftrightarrows_n (\mathfrak{M}', x')$.*

In a picture

$$
\begin{tikzcd}
(\mathfrak{M}, x) \arrow[r, dash, "\leftrightarrow_k"] & (\mathfrak{N}, y) \\
(\mathfrak{M}', x') \arrow[u, "\pi_q"] & (\mathfrak{N}', y') \arrow[u, "\pi_q"'] \arrow[l, "\leftrightarrow_n"]
\end{tikzcd}
$$

Figure 1: Combinatorial condition for uniform definability

In a picture

$$\begin{tikzcd}
(\mathfrak{M}, x) \arrow[r, dash, "\leftrightarrow_k"] & (\mathfrak{N}, y) \\
(\mathfrak{M}', x') \arrow[u, "\pi_q"] & (\mathfrak{N}', y') \arrow[u, "\pi_q"'] \arrow[l, "\leftrightarrow_n"]
\end{tikzcd}$$

Figure 1: Combinatorial condition for uniform definability

It kind of looks like amalgamation. The details of how precisely this is connected are a mystery to everyone.

We show the uniform definability for modal logic $\mathbf{K}$. As we will see, this exploits the rigid structure of $\mathbf{K}$-bisimulations:

Lemma (Combinatorial Lemma)

Let $(\mathfrak{M}, x)$ *be a finite model over* $\overline{p}$ *such that* $(\mathfrak{M}, x) \leftrightarrows (\mathfrak{M}', x')^q$, *and* $(\mathfrak{M}, x) \leftrightarrows_n (\mathfrak{N}, y)$. *Then there is some* $(\mathfrak{N}', y')$ *over* $(\overline{p}, q)$ *such that* $(\mathfrak{N}', y')^q \leftrightarrows (\mathfrak{N}, y)$ *and* $(\mathfrak{N}', y') \leftrightarrows_n (\mathfrak{M}', x')$.

(See the board)

Suppose that the three models in case are the ones from Figure 2.

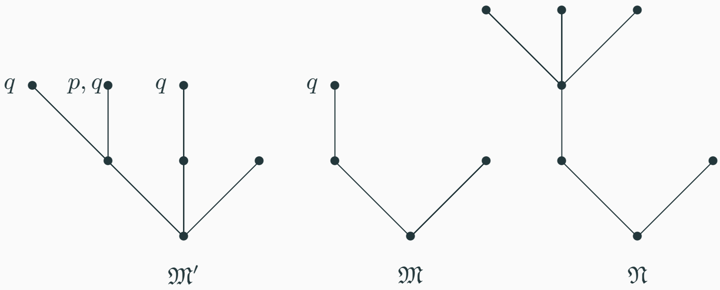

Figure 2: The models $\mathfrak{M}$, $\mathfrak{M}'$ and $\mathfrak{N}$

Example of the construction (cont.d)

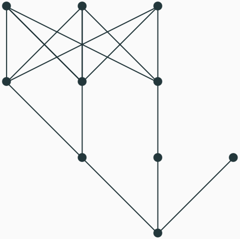

Figure 3: The witness to the combinatorial lemma

The above proof can be adapted for some simple modal systems – $\mathbf{KB}$ and $\mathbf{KD}$ for example – but fails for $\mathbf{S4}$. This is where the equivalences become very useful.

Other modal logics

The above proof can be adapted for some simple modal systems – $\mathbf{KB}$ and $\mathbf{KD}$ for example – but fails for $\mathbf{S4}$. This is where the equivalences become very useful.

The case of other logics – $\mathbf{Grz}$, $\mathbf{GL}$, etc – also goes through with some non-trivial modifications. Similar for $\mathbf{IPC}$.

Uniform deductive interpolation

Uniform deductive interpolation

Definition

A logic L has *uniform deductive interpolation* if and only if whenever $\phi(\overline{p}, \overline{r}) \vdash_L \psi(\overline{q}, \overline{r})$, there two formulas χ_0 and χ_1 in the common language such that:

1. χ_i are deductive interpolants for $\phi \vdash_L \psi$;
2. Whenever μ is a deductive interpolant, then $\chi_0 \vdash \mu$ and $\mu \vdash \chi_1$.

Uniform deductive interpolation

Definition

A logic L has *uniform deductive interpolation* if and only if whenever $\phi(\overline{p}, \overline{r}) \vdash_L \psi(\overline{q}, \overline{r})$, there two formulas χ_0 and χ_1 in the common language such that:

1. χ_i are deductive interpolants for $\phi \vdash_L \psi$;
2. Whenever μ is a deductive interpolant, then $\chi_0 \vdash \mu$ and $\mu \vdash \chi_1$.

As it can be expected, these notions are interrelated:

Definition

Let L be a logic. We say that L has a *deduction theorem* if there is a term $t(x)$ such that for each formulas ϕ, ψ, we have

$$\phi \vdash_L \psi \iff \vdash_L t(\phi) \to \psi.$$

Proposition

Let L be a logic with a deduction theorem. Then if L has uniform Craig interpolation, then L has uniform deductive interpolation.

The theory of uniform deductive interpolation has been extensively developed by Ghilardi, Metcalfe and others.

The theory of uniform deductive interpolation has been extensively developed by Ghilardi, Metcalfe and others.

This has a strongly model-theoretic flavour.

Model completions

The theory of uniform deductive interpolation has been extensively developed by Ghilardi, Metcalfe and others.

This has a strongly model-theoretic flavour.

Definition

Let T be a universal first-order theory. We write $T_\forall$ for the set of universal consequences of T. We say that a theory U is a *cotheory of* T if $U_\forall = T_\forall$. Equivalently, every model of T can be extended to a model of U, and every model of U can be extended to a model of T.
We say that a theory T^* is a *model companion* of T if T and T^* are cotheories and T^* is model-complete.

Definition

Let T be a theory and let T^* be its model companion. We say that T^* is a *model completion* if for every model $\mathfrak{M} \models T$, $T^* \cup Diag(\mathfrak{M})$ is complete.

Definition

Let T be a theory and let T^* be its model companion. We say that T^* is a *model completion* if for every model $\mathfrak{M} \models T$, $T^* \cup Diag(\mathfrak{M})$ is complete.

Proposition

Let T^* be a model companion of T, a theory axiomatised by $\forall\exists$ axioms. The following are equivalent:

1. *T^* is a model completion of T;*
2. *T has the amalgamation property.*

Examples

Let T be the theory of Boolean algebras. Then one can show that T has a model completion.

Let T be the theory of Boolean algebras. Then one can show that T has a model completion.

The theory of T is the theory of atomless Boolean algebras.

Proposition

Let L be a logic. If L has an algebraic model completion, then L has uniform deductive interpolation.

Notably, the converse of this is also true:

Proposition

Let L be a logic which has uniform deductive interpolation. Then L has an algebraic model completion.

Definition

Let $\mathcal{K}$ be a variety of algebras. We say that $\mathcal{K}$ is *coherent* if whenever $\mathcal{A}$ is a finitely presented algebra, and $\mathcal{B} \leqslant \mathcal{A}$ is a finitely generated subalgebra, then $\mathcal{B}$ is finitely presented as well.

Definition

Let $\mathcal{K}$ be a variety of algebras. We say that $\mathcal{K}$ is *coherent* if whenever $\mathcal{A}$ is a finitely presented algebra, and $\mathcal{B} \leqslant \mathcal{A}$ is a finitely generated subalgebra, then $\mathcal{B}$ is finitely presented as well.

Theorem

Let L be a modal logic such that $\mathsf{Alg}(L)$ *is coherent. Then L has a deduction theorem. Moreover, if L has uniform deductive interpolation, then* $\mathsf{Alg}(L)$ *is coherent.*

Definition

Let $\mathcal{K}$ be a variety of algebras. We say that $\mathcal{K}$ is *coherent* if whenever $\mathcal{A}$ is a finitely presented algebra, and $\mathcal{B} \leqslant \mathcal{A}$ is a finitely generated subalgebra, then $\mathcal{B}$ is finitely presented as well.

Theorem

Let L be a modal logic such that $\mathsf{Alg}(L)$ *is coherent. Then L has a deduction theorem. Moreover, if L has uniform deductive interpolation, then* $\mathsf{Alg}(L)$ *is coherent.*

Consequently $\mathbf{K}$ does not have a model completion.

Next time

- Maksimova's characterization of the seven superintuitionistic logics with Craig interpolation.

Next time

- Maksimova's characterization of the seven superintuitionistic logics with Craig interpolation.
- The end (?)

Thank you!
Questions?