Topics in Algebraic Logic and Duality Theory
Lecture 4

Rodrigo N. Almeida, Simon Lemal

June 13, 2025

Plan for the Day

- Maksimova's characterisation of superintuitionistic logics with the interpolation property

Maksimova's characterisation

A superintuitionistic logic has the interpolation property iff it is one of

$$
\begin{aligned}
&\mathsf{CPC}, \\
\mathsf{LC}_2 &= \mathsf{IPC} + \mathsf{bd}_2 + \mathsf{bw}_1 &&= \text{logic of the } 2\text{-chain}, \\
\mathsf{LC} &= \mathsf{IPC} + p \to q \vee q \to p &&= \text{logic of chains}, \\
\mathsf{KC} &= \mathsf{IPC} + \neg p \vee \neg\neg p &&= \text{logic of directed frames}, \\
\mathsf{BD}_2 &= \mathsf{IPC} + \mathsf{bd}_2 &&= \text{logic of frames of depth at most } 2, \\
\mathsf{BD}_2\mathsf{W}_2 &= \mathsf{BD}_2 + \mathsf{bw}_2 &&= \text{logic of the } 2\text{-fork}, \\
&\mathsf{IPC}.
\end{aligned}
$$

Understanding the statement

Definition

We define a sequence of formulas bd_n inductively:

$$\begin{aligned} \mathsf{bd}_0 &= \bot, \\ \mathsf{bd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n), \end{aligned}$$

Logics of bounded depth

Definition

We define a sequence of formulas bd_n inductively:

$$\begin{aligned} \mathsf{bd}_0 &= \bot, \\ \mathsf{bd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n), \end{aligned}$$

Proposition

A frame validates bd_n iff it doesn't contain a chain of $n+1$ points, i.e. iff its depth is bounded by n.

Logics of bounded depth

Definition

We define a sequence of formulas bd_n inductively:

$$\mathsf{bd}_0 = \bot,$$
$$\mathsf{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n),$$

Proposition

A frame validates bd_n iff it doesn't contain a chain of $n+1$ points, i.e. iff its depth is bounded by n.

Proof.

Exercise. $\square$

Logics of bounded width

Definition

We define formulas bw_n as follows:

$$\mathsf{bw}_n = \bigvee_{i=0}^{n} \left(p_i \to \bigvee_{j \neq i} p_j \right).$$

Logics of bounded width

Definition

We define formulas bw_n as follows:

$$\mathsf{bw}_n = \bigvee_{i=0}^{n} \left(p_i \to \bigvee_{j \neq i} p_j \right).$$

Proposition

A frame validates bw_n iff none of its rooted subframes contain an antichain of $n+1$ points, i.e. iff its width is bounded by n.

Definition

We define formulas bw_n as follows:

$$\mathsf{bw}_n = \bigvee_{i=0}^{n} \left(p_i \to \bigvee_{j \neq i} p_j \right).$$

Proposition

A frame validates bw_n iff none of its rooted subframes contain an antichain of $n+1$ points, i.e. iff its width is bounded by n.

Proof.

Exercise. $\square$

Preliminary tools

Jankov formulas

Theorem (Jankov)

Let F be a finite rooted frame (i.e. the dual of a finite SI algebra). Then there is a formula $\chi(F)$ such that for any frame G, we have $G \not\models \chi(F)$ iff F is a p-morphic image of a subframe of G.

Jankov formulas

Theorem (Jankov)

Let F be a finite rooted frame (i.e. the dual of a finite SI algebra). Then there is a formula $\chi(F)$ such that for any frame G, we have $G \not\models \chi(F)$ iff F is a p-morphic image of a subframe of G.

Corollary

Let F be a finite rooted frame and L be a logic. We have $\chi(F) \notin L$ iff $F \models L$.

Jankov formulas

Theorem (Jankov)

Let F be a finite rooted frame (i.e. the dual of a finite SI algebra). Then there is a formula $\chi(F)$ such that for any frame G, we have $G \not\models \chi(F)$ iff F is a p-morphic image of a subframe of G.

Corollary

Let F be a finite rooted frame and L be a logic. We have $\chi(F) \notin L$ iff $F \models L$.

Proof.

Exercise. $\square$

Jankov formulas

Theorem (Jankov)

Let F be a finite rooted frame (i.e. the dual of a finite SI algebra). Then there is a formula $\chi(F)$ such that for any frame G, we have $G \not\models \chi(F)$ iff F is a p-morphic image of a subframe of G.

Corollary

Let F be a finite rooted frame and L be a logic. We have $\chi(F) \notin L$ iff $F \models L$.

Proof.

Exercise. $\square$

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Definition

We turn the class of finite rooted frames into a poset by setting $F \leqslant G$ iff G is a p-morphic image of a generated subframe of F.

Lattice of finite rooted frames

Definition

We turn the class of finite rooted frames into a poset by setting $F \leqslant G$ iff G is a p-morphic image of a generated subframe of F.

Definition

A frame F *covers* a frame G iff $F \leqslant G$ and there is no frame H such that $F < H < G$.

Lattice of finite rooted frames

Definition

We turn the class of finite rooted frames into a poset by setting $F \leqslant G$ iff G is a p-morphic image of a generated subframe of F.

Definition

A frame F *covers* a frame G iff $F \leqslant G$ and there is no frame H such that $F < H < G$.

Proposition

Let F be a finite rooted frame. Then

$$Log(F) = \mathsf{IPC} + \{\chi(G) \mid G \not\geqslant F\}.$$

Poset of finite rooted frames

See board.

Maksimova’s characterisation

Proof of Maksimova’s characterisation

$\Leftarrow$: Exercise.

$\Leftarrow$: Exercise.

$\Rightarrow$: Assume that L is a superintuitionistic with interpolation.

Proof of Maksimova's characterisation

$\Leftarrow$: Exercise.

$\Rightarrow$: Assume that L is a superintuitionistic with interpolation. If $L = \mathsf{CPC}$, we are done.

$\Leftarrow$: Exercise.

$\Rightarrow$: Assume that L is a superintuitionistic with interpolation. If $L = \mathsf{CPC}$, we are done. Otherwise, observe that CPC is the logic of the 1-frame, whose only cover is the 2-chain.

(a) The 1-frame

(b) The 2-chain

Proof of Maksimova's characterisation

$\Leftarrow$: Exercise.

$\Rightarrow$: Assume that L is a superintuitionistic with interpolation. If $L = \mathsf{CPC}$, we are done. Otherwise, observe that CPC is the logic of the 1-frame, whose only cover is the 2-chain.

(a) The 1-frame

(b) The 2-chain

Therefore CPC is axiomatised by the Jankov formula χ_{2c} of the 2-chain.

Proof of Maksimova's characterisation

$\Leftarrow$: Exercise.

$\Rightarrow$: Assume that L is a superintuitionistic with interpolation. If $L = \mathsf{CPC}$, we are done. Otherwise, observe that CPC is the logic of the 1-frame, whose only cover is the 2-chain.

(a) The 1-frame

(b) The 2-chain

Therefore CPC is axiomatised by the Jankov formula χ_{2c} of the 2-chain. As $L \subsetneq \mathsf{CPC}$, we have $\chi_{2c} \notin L$, thus the 2-chain is an L-frame.

If L is the logic of the 2-chain, then $L = \mathsf{LC}_2$, and we are done.

Proof of Maksimova's characterisation (cont.d)

Otherwise, observe that the only covers of the 2-chain are the 3-chain and the 2-fork.

(a) 3-chain

(b) 2-fork

Otherwise, observe that the only covers of the 2-chain are the 3-chain and the 2-fork.

(a) 3-chain

(b) 2-fork

Therefore, LC_2 is axiomatized by the Jankov formulas χ_{3c} of the 3-chain and χ_{2f} of the 2-fork.

Otherwise, observe that the only covers of the 2-chain are the 3-chain and the 2-fork.

(a) 3-chain

(b) 2-fork

Therefore, LC_2 is axiomatized by the Jankov formulas χ_{3c} of the 3-chain and χ_{2f} of the 2-fork. As $L \subsetneq \mathsf{L}_2$, we have $\chi_{3c} \notin L$ or $\chi_{2f} \notin L$, thus either the 3-chain or the 2-fork are L-frames.

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame.

Proof of Maksimova's characterisation (cont.d)

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame. Then $L \supseteq \mathsf{BD}_2 = \mathsf{IPC} + \chi_{3c}$.

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame. Then $L \supseteq \mathsf{BD}_2 = \mathsf{IPC} + \chi_{3c}$. What's more, the 2-fork is an L-frame, thus $L \subseteq \mathsf{BD}_2\mathsf{W}_2$.

Proof of Maksimova's characterisation (cont.d)

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame. Then $L \supseteq \mathsf{BD}_2 = \mathsf{IPC} + \chi_{3c}$. What's more, the 2-fork is an L-frame, thus $L \subseteq \mathsf{BD}_2\mathsf{W}_2$. If $L = \mathsf{BD}_2\mathsf{W}_2$, we are done.

Proof of Maksimova's characterisation (cont.d)

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame. Then $L \supseteq \mathsf{BD}_2 = \mathsf{IPC} + \chi_{3c}$. What's more, the 2-fork is an L-frame, thus $L \subseteq \mathsf{BD}_2\mathsf{W}_2$. If $L = \mathsf{BD}_2\mathsf{W}_2$, we are done. Otherwise, notice that the only cover of the 2-fork which does not cover the 3-chain is the 3-fork.

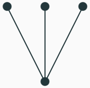

Figure 3: 3-fork

Proof of Maksimova's characterisation (cont.d)

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame. Then $L \supseteq \mathsf{BD}_2 = \mathsf{IPC} + \chi_{3c}$. What's more, the 2-fork is an L-frame, thus $L \subseteq \mathsf{BD}_2\mathsf{W}_2$. If $L = \mathsf{BD}_2\mathsf{W}_2$, we are done. Otherwise, notice that the only cover of the 2-fork which does not cover the 3-chain is the 3-fork.

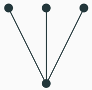

Figure 3: 3-fork

Therefore, BD_2 is axiomatised relative to BD_2 by the Jankov formula χ_{3f} of the 3-fork.

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame. Then $L \supseteq \mathsf{BD}_2 = \mathsf{IPC} + \chi_{3c}$. What's more, the 2-fork is an L-frame, thus $L \subseteq \mathsf{BD}_2\mathsf{W}_2$. If $L = \mathsf{BD}_2\mathsf{W}_2$, we are done. Otherwise, notice that the only cover of the 2-fork which does not cover the 3-chain is the 3-fork.

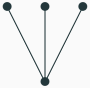

Figure 3: 3-fork

Therefore, BD_2 is axiomatised relative to BD_2 by the Jankov formula χ_{3f} of the 3-fork. As $\mathsf{BD}_2 \subseteq L \subsetneq \mathsf{BD}_2\mathsf{W}_2$, we have $\chi_{3f} \notin L$, thus the 3-fork is an L-frame.

Proof of Maksimova's characterisation (cont.d)

Assume that $\chi_{3c} \in L$, i.e. the 3-chain is not an L-frame. Then $L \supseteq \mathsf{BD}_2 = \mathsf{IPC} + \chi_{3c}$. What's more, the 2-fork is an L-frame, thus $L \subseteq \mathsf{BD}_2\mathsf{W}_2$. If $L = \mathsf{BD}_2\mathsf{W}_2$, we are done. Otherwise, notice that the only cover of the 2-fork which does not cover the 3-chain is the 3-fork.

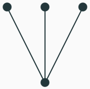

Figure 3: 3-fork

Therefore, BD_2 is axiomatised relative to BD_2 by the Jankov formula χ_{3f} of the 3-fork. As $\mathsf{BD}_2 \subseteq L \subsetneq \mathsf{BD}_2\mathsf{W}_2$, we have $\chi_{3f} \notin L$, thus the 3-fork is an L-frame. An amalgamation argument shows that every n-fork is an L-frame, thus $L = \mathsf{BD}_2$.

Proof of Maksimova's characterisation (cont.d)

Assume that both the 2-fork and the 3-chain are L-frames.

Proof of Maksimova's characterisation (cont.d)

Assume that both the 2-fork and the 3-chain are L-frames. A non-trivial amalgamation trick shows that the 3-fork is an L-frame, thus all n-forks are L-frames.

Assume that both the 2-fork and the 3-chain are L-frames. A non-trivial amalgamation trick shows that the 3-fork is an L-frame, thus all n-forks are L-frames. An amalgamation argument show that every n-chain is an L-frame.

Proof of Maksimova's characterisation (cont.d)

Assume that both the 2-fork and the 3-chain are L-frames. A non-trivial amalgamation trick shows that the 3-fork is an L-frame, thus all n-forks are L-frames. An amalgamation argument show that every n-chain is an L-frame. Another non-trivial amalgamation trick shows that every n-ary tree is an L-frame, thus every finite tree is an L-frame.

Proof of Maksimova's characterisation (cont.d)

Assume that both the 2-fork and the 3-chain are L-frames. A non-trivial amalgamation trick shows that the 3-fork is an L-frame, thus all n-forks are L-frames. An amalgamation argument show that every n-chain is an L-frame. Another non-trivial amalgamation trick shows that every n-ary tree is an L-frame, thus every finite tree is an L-frame. As IPC is the logic of finite trees, we have $L = \mathsf{IPC}$.

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame.

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame. Then $L \supseteq \mathsf{KC} = \mathsf{IPC} + \chi_{2f}$.

Proof of Maksimova's characterisation (cont.d)

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame. Then $L \supseteq \mathsf{KC} = \mathsf{IPC} + \chi_{2f}$. What's more, the 3-chain is an L-frame, and an amalgamation argument shows that every n-chain is an L-frame.

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame. Then $L \supseteq \mathsf{KC} = \mathsf{IPC} + \chi_{2f}$. What's more, the 3-chain is an L-frame, and an amalgamation argument shows that every n-chain is an L-frame. Thus $L \subseteq \mathsf{LC}$.

If $L = \mathsf{LC}$, we are done.

Proof of Maksimova's characterisation (cont.d)

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame. Then $L \supseteq \mathsf{KC} = \mathsf{IPC} + \chi_{2f}$. What's more, the 3-chain is an L-frame, and an amalgamation argument shows that every n-chain is an L-frame. Thus $L \subseteq \mathsf{LC}$.

If $L = \mathsf{LC}$, we are done. Otherwise, observe that LC is axiomatised relative to KC by the Jankov formula χ_d of the diamond, i.e. $\mathsf{LC} = \mathsf{KC} + \chi_d$.

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame. Then $L \supseteq \mathsf{KC} = \mathsf{IPC} + \chi_{2f}$. What's more, the 3-chain is an L-frame, and an amalgamation argument shows that every n-chain is an L-frame. Thus $L \subseteq \mathsf{LC}$.

If $L = \mathsf{LC}$, we are done. Otherwise, observe that LC is axiomatised relative to KC by the Jankov formula χ_d of the diamond, i.e. $\mathsf{LC} = \mathsf{KC} + \chi_d$. As $\mathsf{KC} \subseteq L \subsetneq \mathsf{LC}$, we have $\chi_d \notin L$, so the diamond is an L-frame.

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame. Then $L \supseteq \mathsf{KC} = \mathsf{IPC} + \chi_{2f}$. What's more, the 3-chain is an L-frame, and an amalgamation argument shows that every n-chain is an L-frame. Thus $L \subseteq \mathsf{LC}$.

If $L = \mathsf{LC}$, we are done. Otherwise, observe that LC is axiomatised relative to KC by the Jankov formula χ_d of the diamond, i.e. $\mathsf{LC} = \mathsf{KC} + \chi_d$. As $\mathsf{KC} \subseteq L \subsetneq \mathsf{LC}$, we have $\chi_d \notin L$, so the diamond is an L-frame.

The 3-chain and diamond are L-frames, we proceed as in the case of the 3-chain and the 2-fork, showing that every tree with a point on top is an L-frame.

Proof of Maksimova's characterisation (cont.d)

First assume that $\chi_{2f} \in L$, i.e. the 2-fork is not an L-frame. Then $L \supseteq \mathsf{KC} = \mathsf{IPC} + \chi_{2f}$. What's more, the 3-chain is an L-frame, and an amalgamation argument shows that every n-chain is an L-frame. Thus $L \subseteq \mathsf{LC}$.

If $L = \mathsf{LC}$, we are done. Otherwise, observe that LC is axiomatised relative to KC by the Jankov formula χ_d of the diamond, i.e. $\mathsf{LC} = \mathsf{KC} + \chi_d$. As $\mathsf{KC} \subseteq L \subsetneq \mathsf{LC}$, we have $\chi_d \notin L$, so the diamond is an L-frame.

The 3-chain and diamond are L-frames, we proceed as in the case of the 3-chain and the 2-fork, showing that every tree with a point on top is an L-frame. This way we obtain every directed finite poset, so L must be KC.

Appendix

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$.

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Suppose that X is finite.

Locally finite logics and Jankov formulas

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Suppose that X is finite. If $X \models L$, then $\phi \notin L$.

Locally finite logics and Jankov formulas

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Suppose that X is finite. If $X \models L$, then $\phi \notin L$. If $X \not\models L$, the $\chi(X) \in L'$, so $X \models \chi(X)$, which is absurd.

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\vDash L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\vDash L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Suppose that X is finite. If $X \vDash L$, then $\phi \notin L$. If $X \not\vDash L$, the $\chi(X) \in L'$, so $X \vDash \chi(X)$, which is absurd.

If X is infinite, m-generated, it has infinite depth[a].

Locally finite logics and Jankov formulas

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Suppose that X is finite. If $X \models L$, then $\phi \notin L$. If $X \not\models L$, the $\chi(X) \in L'$, so $X \models \chi(X)$, which is absurd.

If X is infinite, m-generated, it has infinite depth[a]. Let k be the largest depth of an m-generated L model.

Locally finite logics and Jankov formulas

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Suppose that X is finite. If $X \models L$, then $\phi \notin L$. If $X \not\models L$, the $\chi(X) \in L'$, so $X \models \chi(X)$, which is absurd.

If X is infinite, m-generated, it has infinite depth[a]. Let k be the largest depth of an m-generated L model. Take some $x \in X$ at depth $k+1$ and let Y be the subframe generated by x.

Locally finite logics and Jankov formulas

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

$\supseteq$: Exercise.

$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.

Suppose that X is finite. If $X \models L$, then $\phi \notin L$. If $X \not\models L$, the $\chi(X) \in L'$, so $X \models \chi(X)$, which is absurd.

If X is infinite, m-generated, it has infinite depth[a]. Let k be the largest depth of an m-generated L model. Take some $x \in X$ at depth $k+1$ and let Y be the subframe generated by x. Then $Y \not\models L$, so $\chi(Y) \in L'$.

Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(F) \mid F \not\models L\}.$$

Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.
$\supseteq$: Exercise.
$\subseteq$: Suppose $\phi \notin L'$. Let A be a subdirectly irreducible finitely generated model of L' which refutes ϕ and let X be its dual.
Suppose that X is finite. If $X \models L$, then $\phi \notin L$. If $X \not\models L$, the $\chi(X) \in L'$, so $X \models \chi(X)$, which is absurd.
If X is infinite, m-generated, it has infinite depth[a]. Let k be the largest depth of an m-generated L model. Take some $x \in X$ at depth $k + 1$ and let Y be the subframe generated by x. Then $Y \not\models L$, so $\chi(Y) \in L'$. As Y is an L'-frame, we have $Y \models \chi(Y)$, which is a contradiction. □

[a] The curious student should refer to Nick's thesis, sections 3.1 & 3.2

Thank you!
Questions?