Exercise Sheet 1

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1 Set Theory

Exercise 1.1. The following results are used often in topology: given two sets X, Y, a function $f : X \to Y$, sequences of subsets $\{S_i\}_{i\in I} \subseteq \mathcal{P}X$, $\{T_i\}_{i\in I} \subseteq \mathcal{P}Y$ and subsets $S \subseteq X$, $T \subseteq Y$, we have that

1. $f[\bigcup_{i\in I} S_i] = \bigcup_{i\in I} f[S_i]$.
2. $f[\bigcap_{i\in I} T_i] \subseteq \bigcap_{i\in I} f[T_i]$.
3. $f^{-1}[\bigcup_{i\in I} T_i] = \bigcup_{i\in I} f^{-1}[T_i]$.
4. $f^{-1}[\bigcap_{i\in I} T_i] = \bigcap_{i\in I} f^{-1}[T_i]$.
5. $f[S] \cap T = f[S \cap f^{-1}[T]]$

Furthermore, if f is injective then 2 is an equality.

Prove these identities.

2 Basic Topology

Exercise 2.1.

1. Prove that the real line topology is really a topology.
2. Prove that the topology defined over the Cantor set is really a topology.

Exercise 2.2. Let $(\tau_i)_{i\in I}$ be a collection of topologies on a set X.

(a) Is their intersection $\bigcap_{i\in I} \tau_i$ (necessarily) a topology on X?

(b) Is their union $\bigcup_{i\in I} \tau_i$ (necessarily) a topology on X?

(c) Show that there is a greatest topology τ on X such that $\tau \subseteq \tau_i$ for all $i \in I$. (with "greatest" we mean that if τ' is some other topology such that $\tau' \subseteq \tau_i$ for all $i \in I$, then $\tau' \subseteq \tau$)

(d) Show that there is a least topology τ on X such that $\tau_i \subseteq \tau$ for all $i \in I$.

Now let $X = \{x, y, z\}$, $\tau_0 = \{\varnothing, X, \{x\}, \{x, y\}\}$ and $\tau_1 = \{\varnothing, X, \{x\}, \{y, z\}\}$.

(e) Find the greatest topology τ on X such that $\tau \subseteq \tau_0$ and $\tau \subseteq \tau_1$.

(f) Find the least topology τ on X such that $\tau_0 \subseteq \tau$ and $\tau_1 \subseteq \tau$.

3 Closures, Interiors and Neighbourhoods

Definition 3.1. Let (X, τ) be a topological space. We say that a set $U \in \mathcal{P}(X)$ is *closed* if its complement is open; i.e., if $(X - U) \in \tau$. $\dashv$

Definition 3.2. Let (X, τ) be a topological space and $S \subseteq X$ arbitrary. We denote by $cl(S)$ or $\overline{S}$ the *closure of* S, the smallest closed set K such that $S \subseteq K$; that is, $cl(S)$ is the intersection of all closed sets containing S. We denote by $int(S)$ the *interior of* S, the largest open set K such that $K \subseteq S$; that is, $int(S)$ is the union of all open sets contained in S. $\dashv$

Remark 3.1. Using this definition, we have that a set S is closed if and only if $S = \overline{S}$, and open if and only if $S = int(S)$. We call the operators

$$int : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto int(S)$$

and

$$cl : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto cl(S)$$

the *topological interior* and *topological closure*, respectively. As the reader will find in the exercises, interior and closure operators provide an alternative, but equivalent, form of describing topologies. $\dashv$

Definition 3.3. Given a topological space (X, τ) and a point $x \in X$, we say that $V \subseteq X$ is a *neighbourhood* of x if and only if there is an open set U such that $x \in U \subseteq V$.

Moreover, observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x if and only if $x \in V$ and V is open.[1] $\dashv$

Define $N(x) = \{U \in \tau \mid x \in U\}$.

Remark 3.2. (*Epistemic intuition: what is an (open) neighbourhood?*) The open neighbourhoods of a point x have a neat epistemic interpretation: they are precisely the verifiable propositions true at world x (i.e., the propositions that in fact can be verified at x – assuming that only true propositions can be verified). One can also come up with an epistemic interpretation of a neighbourhood simpliciter, but it seems a rather artificial concept; all intuitions, including our epistemic one, have their shortcomings. $\dashv$

Proposition 3.4. Suppose X is a topological space and $S \subseteq X$. Then the following are equivalent for a point $x \in X$:

- x is in the closure of S; i.e., $x \in cl(S)$.
- All open neighbourhoods U of x have non-empty intersection with S; i.e., $U \cap S \neq \varnothing$.

Exercise 3.1. Show the following results about neighbourhoods

1. For every $U \in N(x)$, we have that $x \in U$.
2. $N(x)$ is closed under finite intersections - given a finite sequence $\{V_i\}_{i \in I} \subseteq N(x)$, we have that $\bigcap_{i \in I} V_i \in N(x)$.
3. $N(x)$ is an up set - if $U \in N(x)$ and $U \subseteq V$, then $V \in N(x)$.
4. For every $U \in N(x)$, there exists some $V \in N(x)$, such that $V \subseteq U$ and for every $y \in V$, $U \in N(y)$.

A collection of subsets satisfying conditions (1)-(4) is called a filter.

Given a set X with a map $N : X \to \mathcal{PP}X$ such that $N(x)$ is a filter for every $x \in X$, show that N induces a topology on X. (**Hint:** set U to be open if and only if for every $x \in U$ we have that $U \in N(x)$).

[1] In the literature, you will sometimes find that a neighbourhood simpliciter already is required to be open. We do not adopt that convention, but simply speak of 'open neighbourhoods' when needed.

Exercise 3.2. Prove the following identities for the closure and interior operators. For any topology (X, τ) and sets $A, B \subseteq X$, we have that:

- $A \subseteq \text{cl}\, A$ and $\text{int}\, A \subseteq A$ (extensivity and intensivity).
- If $A \subseteq B$, then $\text{cl}\, A \subseteq \text{cl}\, B$ and $\text{int}\, A \subseteq \text{int}\, B$ (monotinicity).
- $\text{cl}\,(\text{cl}\, A) = \text{cl}\, A$ and $\text{int}\,(\text{int}\, A) = \text{int}\, A$ (idempotency).
- $\text{cl}\, A = X \backslash \text{int}\,(X \backslash A)$ and $\text{int}\, A = X \backslash \text{cl}\,(X \backslash A)$ (duality).
- $\text{int}\, A \cap \text{int}\, B = \text{int}\,(A \cap B)$ and $\text{cl}\, A \cup \text{cl}\, B = \text{cl}\,(A \cup B)$.
- $\text{int}\, A \cup \text{int}\, B \subseteq \text{int}\,(A \cup B)$ and $\text{cl}\, A \cap \text{cl}\, B \subseteq \text{cl}\,(A \cap B)$.
  - Can you come up with an example where equality does not hold? **Hint:** think of the standard topology on $\mathbb{R}$.

Exercise 3.3. We say that U is regular open if $\text{int}\,\text{cl}\, U = U$. Given two regular open sets U, V and a collection $\{U_i\}_{i \in I}$ of regular opens

- Show that $U \cap V$ is regular open.
- Show that $\text{int}\,\text{cl}\,(U \cup V)$ is regular open.
- Show that $\text{int}\,\text{cl}\,(\bigcup_{i \in I} U_i)$ is regular open.
- For any subset A, we have that $\text{int}\,\text{cl}\,\text{int}\,\text{cl}\, A = \text{int}\,\text{cl}\, A$.

4 Topology and Modal Logic

In Modal Logic you were introduced to the epistemic modal logic **S4**; defined through the following rules:

- If I can verify that φ implies ψ, then if I verify φ, then I verify ψ: $\Box(\varphi \to \psi) \vdash (\Box\varphi \to \Box\psi)$ (K axiom).
- If I can verify φ, then φ is true: $\Box\varphi \vdash \varphi$ (T axiom).
- If I can verify φ, then I can verify my verification of φ: $\Box\varphi \vdash \Box\Box\varphi$ (4 axiom).

Exercise 4.1. 1. Can you think of a topological operator that behaves like $\Box$? **Hint:** look at exercise 3.2.

2. Prove the following identity:

$$\text{int}\,(\neg A \cup B) \subseteq \neg(\text{cl}\, A) \cup \text{int}\, B$$

   (a) Show that for any sets A, B, C, we have that $C \subseteq \neg A \cup B$ if and only if $C \cap A \subseteq B$.

   (b) Show that $\text{int}\,(\neg A \cup B) \cap \text{cl}\, A \subseteq \text{int}\, B$. **Hint:** all the tools you need are in exercise 3.2.

   (c) We interpret $A \to B$ as $\neg A \cup B$ and $\vdash$ as $\subseteq$. Confirm that every topological space is an **S4** system.

You were also introduced to neighbourhood semantics: a neighbourhood frame is a pair $\langle W, \mathcal{N} \rangle$ where W is a set of worlds and $\mathcal{N}$ is a map $W \to \mathcal{P}\mathcal{P}W$. A model is a tuple $\langle W, N, V \rangle$ where V is a valuation function $V : \text{At} \to \mathcal{P}W$. We write $[\![p]\!] := V(p) := \{w \in W \mid M, w \vDash p\}$. A formula φ is interpreted as $[\![\varphi]\!] = \{w \in W \mid M, w \vDash \varphi\}$ and $\Box$ is interpreted as $\Box\varphi = \{w \in W \mid \mathcal{N}w[\![\varphi]\!]\}$.

Exercise 4.2. 1. Verify that every topological space is a neighbourhood frame (what is $\mathcal{N}$?)

2. Verify that the topological $\Box$ operator coincides with the neighbourhood semantics $\Box$ operator on that neighbourhood.

Example 4.1. Assume that **S4** is valid on the frame $\langle W, \mathcal{N} \rangle$ but the frame does not satisfy condition 3.1(1). That is, there exists $w \in W$ and $U \in N(x)$ such that $x \notin U$.

As the frame satisfies **S4**, it satisfies $\Box p \vdash p$.

Pick a proposition p and a model $\mathfrak{M} = \langle W, \mathcal{N}, V \rangle$ such that $V(p) = U$. Then $\mathfrak{M}, x \vdash \Box p$ which implies that $\mathfrak{M}, x \vdash p$. But then $x \in V(p) = U$. Which is a contradiction. Meaning that every **S4** frame satisfies condition 3.1(1).

Exercise 4.3. Show that every **S4** neighbourhood frame is a topological space. **Hint:** use exercise 3.1(1) and look at the example.

- Show that every **S4** neighbourhood frame satisfies condition 3.1(2) **Hint:** use the fact that in $mathbf S4$ we have $\Box p \wedge \Box q \vdash \Box(p \wedge q)$.

- Show that every **S4** neighbourhood frame satisfies condition 3.1(3) **Hint:** use the fact that in $mathbf S4$ we have $\Box(p \wedge q) \vdash \Box p \wedge \Box q$.

- Show that every **S4** neighbourhood frame satisfies condition 3.1(4).