Exercise Sheet 2

January 2024

1 Continuity

Recall that we say that given two topological spaces $f : X \to Y$, a function is *continuous* if for each open set $U \subseteq Y$, $f^{-1}[U]$ is open. We say that it is *open* if for each open set $U \subseteq X$, $f[U]$ is open. We furthermore say that f is *closed* if for each closed set V, $f[V]$ is closed.

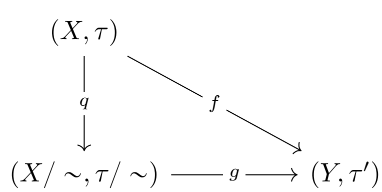
Exercise 1.1. Let $f : X \to Y$ be two partially ordered sets equipped with the Alexandroff topology. Recall that in the lecture it was shown that f is continuous if and only if it is monotone; and it is open if and only if it satisfies the back condition. Give conditions, like those presented in class, for f to be a closed map.

Exercise 1.2. Show the following are equivalent for a map $f : X \to Y$:

1. f is continuous;
2. Inverse images of closed sets are closed;
3. For each set B, $f^{-1}[int(B)] \subseteq int(f^{-1}[B])$.

Definition 1.1. Given a space (X, τ), an equivalence relation $\sim$ on X and the quotient map $q : x \to [x] := \{y \in X \mid x \sim y\}$, we define the quotient topology $\tau/\sim$ on $X/\sim$ as $U \in \tau/\sim$ if $q^{-1}[U] \in \tau$.

Exercise 1.3. Let $(X, \tau), (Y, \tau')$ be topological spaces and $\sim$ an equivalence relation on X. Let $f : (X, \tau) \to (Y, \tau')$ be a continuous map such that for any $x, y \in X$ such that $x \sim y$, we have that $f(x) = f(y)$. Show that there exists a unique continuous map $g : (X/\sim, \tau/\sim) \to (Y, \tau')$ such that $gq = f$.



Exercise 1.4. Consider the subspaces: the interval $I = [0, 1]$ and the unit circle $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$. Consider also the equivalence relation $\sim$ on I defined as $x \sim y$ if $x = y$ or $x, y \in \{0, 1\}$.

Show that $(I/\sim, \tau/sim) \cong S^1$. **Hint:** Show that the map $x \mapsto (\sin 2\pi x, \cos 2\pi x)$ is well defined and induces a homeomorphism.

2 Metric Spaces

Given $\mathbb{R}$ with the standard topology, there is a different definition of continuity which is familiar if you have taken a calculus or analysis course:

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a map. We say that f is *analytically continuous at a point* c if and only if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in \mathbb{R}$, if $|c - x| < \delta$ then $|f(c) - f(x)| < \varepsilon$. A function is analytically continuous if and only if it is analytically continuous at every point c.

Denote: $B^c_r = \{x \in \mathbb{R} \mid |x - c| < r\}$. This is often called the *open ball* around c of radius r. Notice that analytic continuity amounts to for all B^c_δ there exists $B^{f(c)}_\varepsilon$ such that if $x \in B^c_\delta$ then $f(x) \in B^{f(c)}_\varepsilon$.

Exercise 2.1.
1. Given an open set U in the standard topology on the reals, show that for any $x \in U$ there exists ε such that $B^x_\varepsilon \subseteq U$.
2. Show that for a map $f : \mathbb{R} \to \mathbb{R}$ it is analytically continuous if and only if it is continuous with regards to the standard topology on $\mathbb{R}$.

One of the first reasons people have explored topology is to axiomatise the meaning of a metric space

Definition 2.2. A metric space is a pair (X, d) with a set X and a map $d : X \times X \to [0, \infty)$, such that for any $x, y, z \in X$, the following conditions hold:

1. $d(x, x) = 0$.
2. If $x \neq y$, then $d(x, y) > 0$ (separation).
3. $d(x, y) = d(y, x)$ (symmetry).
4. $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality).

Exercise 2.2. Verify that $\mathbb{R}$ with the distance function $|y - x|$ forms a metric space. **Hint:** Show that $|x - y|^2 \leqslant (|x - z| + |z - y|)^2$ and use the fact that for any $a, b > 0$ we have that $a^2 \leqslant b^2$ if and only if $a < b$.

In fact, we also have that $\mathbb{R}^2$ with the Euclidean distance function $d((x_1, y_1), (x_2, y_2)) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$ is a metric space.

Exercise 2.3. We define a ball in a metric space (X, d) as $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Show that the balls are closed under binary intersections and that it covers X. We call this the metric topology.

Notice that on the real line, the metric topology is the standard topology.

Definition 2.3. Given two metric spaces (X, d_1) and (Y, d_2) we define an isometry to be a bijective map $f : (X, d_1) \to (Y, d_2)$ such that for any $x, y \in X$ we have that $d(f(x), f(y)) = d(x, y)$

Exercise 2.4. Show that every isometry is a homeomorphism.

Exercise 2.5. Given a metric space (X, d), a topological space (Y, τ) and a homeomorphism $f : (Y, \tau) \to (X, d)$, show that f induces a metric space on (Y, τ).

3 Filters and Filter Convergence

In class we saw the definition of a filter base and of filter convergence. We briefly recall these definitions here:

Definition 3.1. Let X be a set. We say that a collection of subsets $F \subseteq \mathcal{P}(X) - \{\varnothing\}$ is a *filter base* if it satisfies the following:

- $X \in F$;
- If $A, B \in F$ then $A \cap B \in F$.

We say that a given filter base is a *filter* if it is upwards closed: whenever $A \in F$ and $A \subseteq B$ then $B \in F$.

Solve the exercise from the previous exercise sheet, now with a newfound glee from understanding the definition of a filter:

Exercise 3.1. Show that neighbourhoods are a filter such that:

1. For every $U \in N(x)$, we have that $x \in U$.
2. $N(x)$ is closed under finite intersections - given a finite sequence $\{V_i\}_{i\in I} \subseteq N(x)$, we have that $\bigcap_{i\in I} V_i \in N(x)$.
3. $N(x)$ is an up set - if $U \in N(x)$ and $U \subseteq V$, then $V \in N(x)$.
4. For every $U \in N(x)$, there exists some $V \in N(x)$, such that $V \subseteq U$ and for every $y \in V$, $U \in N(y)$.

A collection of subsets satisfying conditions (1)-(4) is called a filter.

Given a set X with a map $N : X \to \mathcal{P}\mathcal{P}X$ such that $N(x)$ is a filter for every $x \in X$, show that N induces a topology on X. (**Hint:** set U to be open if and only if for every $x \in U$ we have that $U \in N(x)$).

Exercise 3.2. Let $\{a, b, c\}$ be a set with three elements. Describe all filters and filter bases on this set.

Exercise 3.3. Let (X, τ) be a topological space. Show the following:

1. If F is a filter base, then
$$\uparrow F = \{A \subseteq X : \exists B \in F, B \subseteq A\},$$
is a filter.
2. Let $F \subseteq \mathcal{P}(X)$. We say that F has the *finite intersection property* if for all $A_0, ..., A_n \in F$, $A_0 \cap ... \cap A_n \neq \varnothing$. Show that if F has the FIP then
$$F^{\cap} = \{A_0 \cap ... \cap A_n : A_0, ..., A_n \in F\}$$
is a filter base.

Definition 3.2. Given a topological space (X, τ) and a filter F on X, we say that F is a convergent filter to some point $x \in X$ if for any $U \in N(x)$ we have that $U \in F$.

Exercise 3.4. Consider the subspace $[0, 1]$ of $\mathbb{R}$. We call a sequence $\{a_n : n \in \omega\}$ a *converging sequence* to a limit x if for any $U \in \tau$, $x \in U$ implies that there exists a natural number N such that for any $n > N$ we have that $a_n \in U$.

- Show that in $[0, 1]$, the limit is unique. That is, if a sequence converges, it converges to a unique point x. **Hint:** assume the opposite and construct two open sets that contradict this assumption.
- Show that each converging sequence defines a converging filter, and each converging filter contains a converging sequence.

Exercise 3.5. Describe converging filters over the Cantor space.

4 The Topology Poem

Let's get down to business, to defeat confusion, Did they send me students, when I asked for solutions? You're the math-iest bunch I ever met, But you can bet, before we're through, Students, I'll make a topologist out of you.

Tranquil as a torus, but on proof within, Once you find your lemma, you are sure to win. You're a jumbled, complex, chaotic lot, And you haven't proved a thing. Somehow I'll make a topologist out of you.

I'm never gonna skip a class, Say goodbye to those easy As, Boy, was I a fool in school for taking gym! This proof's got me scared to death, Hope it doesn't see right through my sketch. Now I really wish that I knew what a "lim" is.

We must be sharp as the Euler's insight, (With logic!) With all the force of a great bijection, (With logic!) With all the strength of a bounded sequence, Mysterious as the dark side of the Möbius strip.

Time is racing toward us, till the finals arrive, Heed my every theorem, and you might survive. You're unsuited for the rage of proof, So pack up, go home, you're through. How could I make a topologist out of you?

We must be sharp as the Euler's insight, (With logic!) With all the force of a great bijection, (With logic!) With all the strength of a bounded sequence, Mysterious as the dark side of the Möbius strip.