Exercise Sheet 3

January 2024

1 Separation Axioms

Separation axioms teach us how separated are points in a space (see lecture notes). The four important ones are

Definition 1.1 (Separation Axioms)**.** Given a topological space (X, τ) we say that

- It is Kolmogorov if given any two points x, y there exists $U \in \tau$ such that either $x \in U \wedge y \notin U$ or $y \in U \wedge x \notin U$. We denote this separation axiom by T_0.
- It is a Frećhet if given any two points x, y there exists $U, V \in \tau$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$. We denote this separation axiom by T_1.
- It is Hausdorff if given any two points x, y there exists $U, V \in \tau$ such that $x \in U, y \in V, U \cap V = \varnothing$. We denote this separation axiom by T_2.
- It is normal if it is T_1 and given any two disjoint closed sets E, K, there exists $U, V \in \tau$ such that $E \subseteq U, K \subseteq V, U \cap V = \varnothing$. We denote this separation axiom by T_4.

Exercise 1.1. Show that a space is Frećhet if and only if every finite subset is closed.

Exercise 1.2. Show that every metric topology is Hausdorff.

Exercise 1.3. Let $\mathcal{T}_n$ represent the collection of topological spaces of separation axiom n. Show that

$$\mathcal{T}_4 \subset \mathcal{T}_2 \subset \mathcal{T}_1 \subset \mathcal{T}_0.$$

Exercise 1.4.
1. Show that for any closed subset K we have that if E is closed in the subspace topology of K, then E is closed in the general topology.
2. Show that a closed subset of a normal space is normal.

Exercise 1.5. Prove that all the separation axioms are topological properties.

Exercise 1.6.
- Given a collection of sets $\{U_i\}_{i \in I}$, an atom is a non-empty set $U_j \neq \varnothing$ such that for any $i \in I$, we have that $U_i \subseteq U_j$ if and only if $U_i = U_j$ or $U_i = \varnothing$.
- Given a topological space (X, τ) and a subset $S \subseteq X$, an isolated point $x \in S$ is a point such that there exists $U \in \mathcal{N}$ with $U \nsubseteq S$. An isolated point of X is an open set of the form $\{x\}$.

Given a Hausdorff space (X, τ) and its collection of regular open sets $\mathcal{U} \subseteq \tau$, show that a regular set A is an atom if and only if A is an isolated point. (**Hint:** assume that this is false and show that for some strict subset $U \subset A$, we have that $\operatorname{int} \operatorname{cl} U \in \mathcal{U}$.)

2 Metrics and Compactness

Exercise 2.1. Given a compact space X and a continuous surjective map $f : X \to Y$, show that Y is compact.

Exercise 2.2. 1. Show that every compact subset of a metric space is closed.

We say that a set S is bounded if there exists a number M such that for any $a, b \in S$ we have that $d(a, b) < M$.

2. Show that every compact subset of a metric space is bounded. **Hint:** think of the unit balls cover of the subset.

3. Show that in $\mathbb{R}^n$ with the standard metric topology, if a set S is bounded, then there exists some ball B_0^r such that $S \subseteq B_0^r$.

4. Given $[a, b] \subseteq \mathbb{R}$, show that $[a, b]$ is compact by showing that for every open cover $\mathcal{U}$, we can show that for $b' = \sup\{x \in [a, b] \mid [a, x]$ can be covered by a finite subset of $\mathcal{U}\}$ we have that $b = b'$.

5. Use Tychonoff's theorem and all your hard work from this exercise to show that every bounded closed subset of $\mathbb{R}^n$ is compact.

3 Locales and Separations

We can view every epistemic propositional system as a poset $(P, \vdash)$ where for any φ, ψ we have that $\varphi \vdash \psi$ if and only if we can prove ψ from φ and the operators $\wedge, \vee, \to, \bigvee, \top, \bot$ behave as expected. That is, for any $\varphi, \psi, \chi \in P$, we have that

$$\chi \vdash \varphi \wedge \psi \Leftrightarrow \chi \vdash \varphi \text{ and } \chi \vdash \psi$$
$$\bigvee_{i \in I} \varphi_i \vdash \chi \Leftrightarrow \text{there exists } i \text{ such that } \varphi_i \vdash \chi$$
$$\chi \vdash \varphi \to \psi \Leftrightarrow \chi \wedge \varphi \vdash \psi$$

A poset with all of these operators is called a frame.

Exercise 3.1. Confirm that for any topological space (X, τ), τ is a frame. I.e. show that there exists operators $\wedge, \vee, \to, \bigvee, \top, \bot$ that behave as expected. $\to$ might be tricky. (**Hint:** Consider $\text{int}\,(U \cup V^C)$)

Exercise 3.2. In intuitionistic logic, we define negation as $\neg - := - \to \bot$. Confirm that in the real line $\mathbb{R}$ with the standard topology there exists a set K such that $K \vee \neg K \neq \top$. (**Hint:** Consider $\mathbb{Q}$ and show that $\text{int}\,\mathbb{Q} = \varnothing$)

Exercise 3.3. In the previous exercise sheet, you learned that regular open sets are closed under finite intersections and arbitrary applications of $\text{int cl} \bigcup$.

Given the standard topology on $\mathbb{R}$, consider the collection of regular opens $\mathcal{U}$. Given any $U, V \in \mathcal{U}$, denote $U \wedge V := U \cap V$, $U \vee V := \text{int cl}\,(U \cup V)$, $U \to V := \text{int cl}\,(\text{int}\,(U^C) \cup V)$ and $\neg U := \text{int}\,(U^C)$.

1. Confirm that this indeed forms a frame. You do not need to show that implication and negation are regular open sets.

2. This frame is actually an example of a complete Boolean algebra. Complete boolean algebras are frames in which $\varphi \vee \neg\varphi = \top$ and there exists an operator $\bigwedge$. Confirm that the frame indeed satisfies both conditions.

In class, you were introduced to filters. A filter over a poset P is an up set $F \subseteq P$ that is closed under intersections. That is: if $\varphi, \psi \in F$ then $\varphi \wedge \psi \in F$ and if $\varphi \in F$ and $\varphi \vdash \psi$ then $\psi \in F$. A proper filter is a filter such that $\bot \notin F$. A complete prime filter is a proper filter F such that given a collection of elements $\{\varphi_i\}_{i \in I}$ such that $\bigvee_{i \in I} \varphi_i \in F$, there exists $i \in I$ such that $\varphi_i \in F$.

A model of a frame A can be fixed as a collection of propositions that behaves as a completely prime filter. That is, it can be considered as a map $\chi : A \to \{\top, \bot\}$ that respects the structure of the frame: $\chi(\varphi \wedge \psi) = \chi(\varphi) \wedge \chi(\psi), \ldots$. A mapping such as φ is called a frame homomorphism.

Exercise 3.4. Given the space $\mathbf{1} : (*, \{\varnothing, *\})$ and a topological space (X, τ), a continuous map $x : \mathbf{1} \to X$ fixes a point in X. Consider the collection of open sets $\{U \in \tau \mid U \subseteq x^{-1}(*)\}$. Verify that this set is a completely prime filter. That is, that x^{-1} is a frame homomorphism.

Denote $\text{pt}\, A = \{\chi : A \to \mathbf{2} \mid \chi \text{ is a frame homomorphism}\}$. For a given $\chi \in \text{pt}\, A$, denote $\chi \vDash \varphi$ if and only if $\chi(\varphi) = \top$. Thus we can see that for a frame A, we can generate a set of models by considering all the frame homomorphism as points. The pair $(\text{pt}\, A, A)$ is called a locale. It is tempting to replace the satisfaction $\vDash$ relation with a membership $\in$ relation. It is also clear from the previous exercise that if a locale is a topological space, then its topology is the elements of the frame itself. That is, $\tau = A$.

Note however that not every frame generates a topology.

Exercise 3.5. In Mathematical Structures in Logic you would learn that in a complete atomic Boolean algebras, every completely prime filter F contains an atomic element $x = \bigwedge F \in F$.

Reason that this implies that the set of points of a regular open frame generated from a Hausdorff topology is empty and so, there is no topology.

A locale that generates a topology is called a spatial locale.

Notice that we have that for any $\varphi, \psi \in A$, if $\varphi \vdash \psi$, then for any $\chi \in \text{pt}\, A$, if $\chi \vDash \varphi$ then $\chi \vDash \psi$. This means that $\text{pt}\, A$ is always sound in relation to the frame.

It can be shown that a locale is spatial if and only if for all $\varphi, \psi \in A$, if for all $\chi \in \text{pt}\, A$ we have that $\chi \vDash \varphi$ implies $\chi \vDash \psi$, then $\varphi \vdash \psi$. That is, a locale is spatial if A is complete with respect to $\text{pt}\, A$.

The separation axioms can be then understood as statements about the models. We say that y specializes x if and only if, for any proposition φ, if $x \vDash \varphi$ then $y \vDash \varphi$. Denote it by $x \sqsubseteq y$.

Exercise 3.6. Confirm that

1. A space is Kolmogorov if and only if the specialization relation is a partial order.

2. A space is Frećhet if and only if $\sqsubseteq$ is the equality over the space.

3. In Hausdorff spaces we have that if $x \sqsubseteq y$ then $x = y$.

Exercise 3.7. Consider the cofinite space over the naturals $(\mathbb{N}, \tau)$.

1. Show that a cofinite set U is a completely prime filter in τ if and only if there exists $n \in \mathbb{N}$ such that $U = \mathbb{N} \backslash \{n\}$.

2. Show that $\omega := \tau \backslash \{\varnothing\}$ is a completely prime filter in τ.

You have shown that $(\text{pt}\, \tau, \tau)$ is the same as $(\mathbb{N} \cup \{\omega\}, \tau)$ where ω is in every open set. Show that this is not homeomorphic to the original topology (**Hint:** recall that the separation axioms are topological properties).

A space that is homeomorphic to its lattice is called sober. A space is sober if and only if for every completely prime filter F there exists a point x such that F is its neighbourhood $\mathcal{N}(x)$.

Another characterisation of sober spaces is as follows: an open set $W \in \tau$ is meet-irreducible if and only if for any $U, V \in \tau$ $U \cap V \subseteq W$ if and only if $U \subseteq W$ or $V \subseteq W$. A space is sober if and only if the only meet-irreducible sets are of the form $X \backslash \text{cl}\, \{x\}$.

Exercise 3.8. Show that all Hausdorff spaces are sober.

Exercise 3.9. Show that every sober space is Kolomogrov.

Tying all the exercises we saw together, we get that sobriety can be treated as its own separation axiom, weaker than Hausdorff but stronger than Kolomogrov. As we saw that not every Frećhet space is sober, we get that sobriety is distinct from Frećhet.