Our last time together :(

January 2024

1 Compactifications

For the purpose of this exercise sheet we will let the notion of "compactification" mean what is called in the slides a "decent compactification".

Exercise 1.1. Let X be a topological space. Let $\mathcal{C}(X)$ be the class of all compactifications of X. Define a relation on this as follows: $(Y_1, c_1) \leqslant (Y_2, c_2)$ if there is a continuous function $f : Y_2 \to Y_1$ such that $fc_2 = c_1$. Show that $\leqslant$ is a preorder relation. Moreover show that if $Y \leqslant Z$ and $Z \leqslant Y$ then there is a homeomorphism between Y and Z.

Exercise 1.2. Let X be the space of $\mathbb{N}$ with the discrete topology.

1. Consider the space $(\mathbb{N} \cup \{\omega\}, \tau)$ where τ is the topology generated by the basis of finite and cofinite sets. Verify that this is a topology and prove that this is homeomorphic to the Alexandroff compactification of the discrete topology over $\mathbb{N}$.

2. Consider the space $\mathbb{N}^* = (\mathbb{N} \cup \{\omega_0, \omega_1\}, \tau^*)$ where τ^* is the topology generated by the following subbasis:

   $$\mathsf{Fin}(\mathbb{N}) \cup \{\{2n : n \in \omega\} \cup \{\omega_0\}\} \cup \{\{2n+1 : n \in \omega\} \cup \{\omega_1\}\}.$$

   Show that this space is a Stone space, and that it is a compactification of $\mathbb{N}$. Also show that:

   $$\alpha(\omega) < \mathbb{N}^* < \beta(\omega)$$

   with respect to the preorder defined in 1.1.

2 Connectedness

Exercise 2.1. Give an example of a path $p : [0,1] \to X$ connecting a to b in the space

$$(\{a, b, c, d\}, \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, d, c\}, X\})$$

([1])

A space (X, τ) is path-connected if a path connects any $x, y \in X$. That is for any $x, y \in X$, there exists a continuous map $p : [0,1] \to X$ such that $p(0) = x$ and $p(y) = 1$.

Lemma 2.1. For any subset B of $\mathbb{R}$ that is bounded above (i.e. there exists some upper bound a of B such that for any $b \in B$ we have that $b \leqslant a$), there exists a real number m such that m is an upper bound of B and for any other upper bound a of B we have that $b \leqslant a$. We call m the supremum.

Exercise 2.2. 1. Show that the connected subsets of $\mathbb{R}$ are the intervals.

2. Show that if X is connected or path-connected and $f : X \to Y$ is a continuous map, then $f[X]$ is connected or path connected.

3. Conclude that every path-connected space is connected.

Lemma 2.2 (Brouwer's fixed point theorem - one dimensional case)**.** For every continuous function $f : [-1,1] \to [-1,1]$, there exists a point $x \in [-1,1]$ such that $f(x) = x$.

Exercise 2.3. Prove Brouwer's fixed point theorem. **Hint:** assume for contradiction that it is false and show how this would imply that the interval is not connected.

Exercise 2.4. The topologist's sine curve is given by the set $(0,0) \cup \{(x, \sin\frac{1}{x}) \mid x > 0\}$. Notice that

$$\begin{aligned} \mathbb{R}^+ &\to (\mathbb{R}^+)^2 \\ x &\mapsto (x, \sin\frac{1}{x}) \end{aligned}$$

is a continuous map.

Show that the topologist's sine curve is connected. Is it path-connected?

3 Homotopies

We say that x is homotopic to y, denoted by $x \sim y$ if they are path-connected.

Exercise 3.1. Show that path connectedness forms an equivalence relation on a space X. That is $\sim$ is a reflexive, symmetric and transitive relation.

Given two functions $f, g : X \to Y$, we say that they are homotopic ($f \sim g$) if there exists a continuous map $H : X \times I \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$.

Exercise 3.2. We call paths $p : [0,1] \to X$ such that $p(0) = p(1) = x$ loops based on x. Show that $\sim$ is an equivalence relation on the collection of loops based on x.

Given two spaces(!) X and Y, we say that X and Y are homotopic ($X \simeq Y$) if there exists a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that $fg \sim Id_Y$ and $gf \sim Id_X$.

Exercise 3.3. 1. Show that $\mathbb{R}^2 \simeq \{0\}$.

2. Show that $\mathbb{R}^2 \backslash \{0\} \simeq S^1$.

Show them what a covering is:

We define a covering to be a map $p : \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X, such that for every α, the preimage is a disjoint union of open sets

$$p^{-1}(U_\alpha) = \bigsqcup V_\alpha^\beta$$

and such that the restriction $p\restriction_{V_\alpha^\beta} : V_\alpha^\beta \to U_\alpha$ is a local homeomorphism.

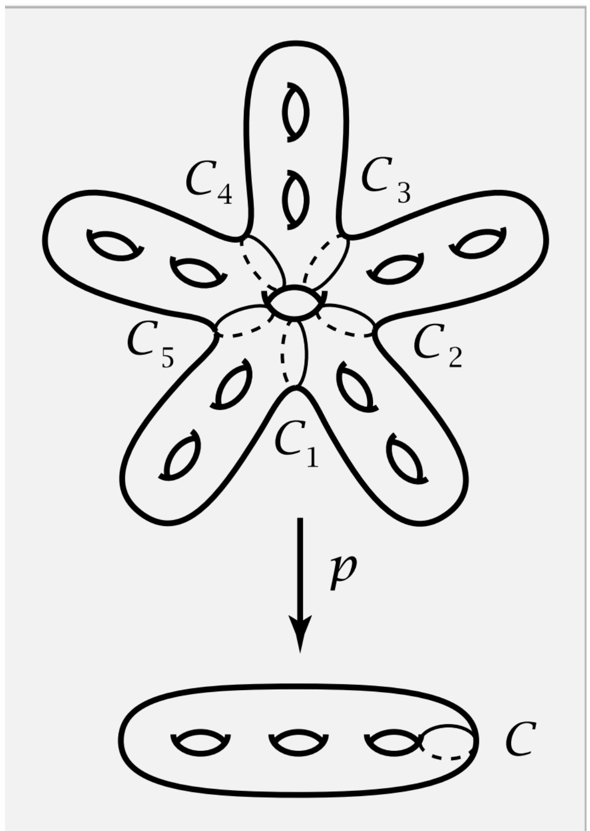
Exercise 3.4. Show that the map

$$\begin{aligned} p_\infty : \mathbb{R} &\to S^1 \\ t &\to (\cos 2\pi t, \sin 2\pi t) \end{aligned}$$

is a covering.

References

[1] T.D. Bradley, T. Bryson, and J. Terilla. *Topology: A Categorical Approach.* MIT Press, 2020.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