Topology project, 1st lecture

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Plan for the day

- A few announcements
- Structure of the lecture
- Actual lecture
- Break from 15h45-16h
- More lecturing

Announcements and structure

Announcements:

- Lecture notes have been published
- First assignment has been published (deadline: Feb 10th)
- First MC quiz to be published today (deadline: Friday)

Structure of the lecture:

- Lecture covers (most of) the content of the lecture notes
- Both slides and blackboard
- @People on Zoom: If you have a question, let Rodrigo know

The concept of a topological space

– Appears abstract and arbitrary

– We will give it a rich epistemic meaning

Definition (topological space)

Let X be a set. $\tau \subseteq \mathcal{P}(X)$ is a *topology on* X :iff

(O1) $\varnothing$ and X are in τ; i.e., $\varnothing \in \tau$ and $X \in \tau$.

(O2) τ is closed under *arbitrary* unions.

(O3) τ is closed under *finite* intersections.

Terminology:

- τ is a *topology* on X
- (X, τ) is a *topological space* (or simply: X is a top. sp.)
- $U \in \tau$ is *open*

Epistemic intuition

Definition (topological space)

Let X be a set. $\tau \subseteq \mathcal{P}(X)$ is a *topology on* X :iff

(O1) $\varnothing$ and X are in τ; i.e., $\varnothing \in \tau$ and $X \in \tau$.

(O2) τ is closed under *arbitrary* unions.

(O3) τ is closed under *finite* intersections.

Epistemic interpretation:

(X) We think of X as a set of 'epistemic worlds'.

(τ) We think of $\tau \subseteq \mathcal{P}(X)$ as corresponding to the set of 'verifiable propositions'.

What is a ‘verifiable proposition’?

Verifiable propositions

What is a proposition?

See Blackboard

What does 'verifiable' mean?

Consider the propositions

($\exists\neg WS$) *There is a non-white swan.*

and its negation

($\forall WS$) *All swans are white.*

Definition

A proposition P is *verifiable* :iff whenever P is true at a world x (i.e., $x \in [\![P]\!]$), it is possible to verify P at x (i.e., verify $x \in [\![P]\!]$).

Let’s check that (O1)-(O3) are satisfied

Epistemic intuition (cont.)

1. *See blackboard for example*
2. **Why this asymmetric definition of a topological space?**
   - Arbitrary intersections? (see blackboard)
   - Complements? (see blackboard)
3. Summary:

Logic	Topology
Epistemic worlds/situations/models/etc.	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$

4. Examples of top. spaces (blackboard):
   - $X = \{x, y, z\}$ with mult. (non-)tops
   - Discrete and indiscrete
   - S4 frames

Basis and subbasis

Definition (basis and subbasis)

Given a top. sp. (X, τ), $\mathcal{B} \subseteq \tau$ is a *basis for the topology* τ :iff $\forall U \in \tau$, $\exists (V_i)_{i \in I} \subseteq \mathcal{B}$ s.t.

$$U = \bigcup_{i \in I} V_i.$$

Further, $\mathcal{S} \subseteq \tau$ is a *subbasis for the topology* :iff $\{\bigcap_{V \in M} V \mid M \subseteq \mathcal{S}, M \textit{is finite}\}$ forms a basis for the topology.

Terminology: Given a (sub)basis $\mathcal{B} \subseteq \tau$, we call members $U \in \mathcal{B}$ *(sub)basic opens*.

Proposition

Let X be a set and $C \subseteq \mathcal{P}(X)$ a collection of sets. Then there is a (unique) topology on X for which C is a subbasis.

Moreover, if (1) C covers X (i.e., $\bigcup_{U \in C} U = X$) and (2) C is closed under finite non-empty intersections, then there is a (unique) topology on X for which C is a basis.

Proof idea

See blackboard.

Epi. int.: what is a (sub)basis?

Examples of (sub)bases

Epistemic example

Let $X = \{a, b, c, d\}$ where a, b, c, d are worlds described as follows:

	All ravens are black	Some raven is non-black
$(\forall WS)$	a	b
$(\exists\neg WS)$	c	d

See blackboard for rest of example

More examples (see blackboard):

- Real line topology
- Cantor and Baire spaces

Comparing topologies

Definition

Let X be a set, and τ and τ' two topologies on this set. We say that τ is a *coarser topology* than τ' if $\tau \subseteq \tau'$. Conversely, we say that τ' is *finer* than τ.

(Highly useful) lemma

Suppose X is a set with two topologies τ and τ', and $\mathcal{B}_\tau$ and $\mathcal{B}_{\tau'}$ are bases for these topologies, respectively. Then $\tau \subseteq \tau'$ iff for all points $x \in X$ and all basic τ-open $U \in \mathcal{B}_\tau$ containing x, there is some basic τ'-open $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$.

Proof

See blackboard.

Comparing tops on $\mathbb{R}$

Euclidean top and top gen. by basis $\{(l, \infty) \mid l \in \mathbb{R}\}$. See blackboard.

Questions?

Generating New Topologies: Subspaces

Definition (subspace)

Let (X, τ) be a ts and $S \subseteq X$. We denote by τ_S the *subspace topology on S* defined as

$$\tau_S := \{U \cap S \mid U \in \tau\}.$$

Terminology: (S, τ_S) is a *subspace* of (X, τ).

Lemma (subspace basis)

Let (X, τ) be a ts with a basis $\mathcal{B}$, and let $S \subseteq X$. Then the set

$$\mathcal{B}_S = \{U \cap S : U \in \mathcal{B}\}$$

is a basis for τ_S.

Proof

See blackboard.

Subspace top on $\mathbb{Z} \subseteq \mathbb{R}$

See blackboard.

Generating New Topologies: Finite Products

Definition (product top)

Let X and Y be ts. We define a topology on the product $X \times Y$, called the *product topology*, as follows: a set $U_0 \times U_1 \subseteq X \times Y$ is basic open :iff U_0 is open in X and U_1 is open in Y.

Proposition (subspaces and products commute)

Suppose X and Y are ts; $S_X \subseteq X$; and $S_Y \subseteq Y$. Then first constructing the product topology $X \times Y$ and then constructing the subspace topology $S_X \times S_Y \subseteq X \times Y$ is the same as first constructing the subspace topologies $S_X \subseteq X$ and $S_Y \subseteq Y$ and then taking their product $S_X \times S_Y$.

Proof

See blackboard.

Lemma

Let X and Y be ts with bases $\mathcal{B}_X$ and $\mathcal{B}_Y$. Then $\{U_X \times U_Y \mid U_X \in \mathcal{B}_X, U_Y \in \mathcal{B}_Y\}$ forms a basis for the product topology on $X \times Y$.

Proof of lemma is an exercise.