Topological semantics of modal logic

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Organization

- 5 lectures + tutorials/Q & A sessions.
- Exercise sheets will be provided each day.
- There will be a written exam at the end of the course.

Literature

- P. Blackburn, M. de Rijke, and Y. Venema, Modal Logic, Cambridge University Press 2001.
- J. van Benthem and G. Bezhanishvili, Modal Logic of Space, Handbook of Spatial Logics, 2007.

Prerequisites

- Basic knowledge of modal logic.
- Familiarity with some basic concepts of general topology.

Content

- Overview of relational semantics of modal logic,
- Topological semantics,
- Topo-bisimulations,
- Topo-canonical models,
- Basic completeness results for topological semantics,
- McKinsey-Tarski theorem,
- Derived set operator semantics,
- Topological semantics of modal fixed-point logic (time permitting).

Logic and topology

- Topology is a mathematical theory of space.
- Logic is a formal theory of reasoning.
- Spatial logic is a logical analysis of space.
- In this course we will concentrate on spatial/topological modal logic.

Modal Logic

Modal Logic = Classical Logic + $\Box$, $\Diamond$.

It is very expressive yet decidable (fragment of the First-Order Logic).

Modal logic admits algebraic, relational and topological semantics.

Topological semantics of modal logic was introduced and developed by McKinsey and Tarski in 1930's and 1940's of the 20th century.

One of the early reference along McKinsey and Tarski is Tang Tsao Chen (1938).

Early references

- T. Tsao-Chen, Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication, *Bulletin of the American Mathematical Society*, vol. 44 (1938), pp. 737-744. (National Wuhan University).
- A. Tarski, Der Aussagenkalkül und die Topologie, *Fundam. Math.* 31 (1938), 103-134.
- J. C. C. McKinsey, A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology, *Journal of Symbolic Logic*, vol. 6 (1941), pp. 117-134.
- J. C. C. McKinsey and A. Tarski, The algebra of topology, *Annals of Mathematics*, vol. 45 (1944), pp. 141-191.

Alfred Tarski

Alfred Tarski (1901 - 1983)

Part 1: Modal logic and its relational semantics

Modal language

Let Prop denote the set of propositional letters.

The language of basic modal logic is defined by the grammar

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi,$$

where $p \in \mathsf{Prop}$

We assume the standard abbreviations: $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Most importantly: $\Box\varphi := \neg\Diamond\neg\psi$.

Kripke semantics

Definition. A Kripke frame is a pair (W, R), where W is a nonempty set and $R \subseteq W^2$ is a binary relation.

A Kripke model is a triple (W, R, V), where (W, R) is a Kripke frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$.

Let $\mathfrak{M} = (W, R, V)$ be a Kripke model and $w \in W$. We defined by induction when a formula φ is satisfied at w in $\mathfrak{M}$, written $\mathfrak{M}, w \models \varphi$:

$\mathfrak{M}, w \models \bot$ never

$\mathfrak{M}, w \models p$ iff $w \in V(p)$,

$\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$,

$\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$,

$\mathfrak{M}, w \models \Diamond\varphi$ iff $\exists v \in W$ such that wRv and $\mathfrak{M}, v \models \varphi$.

$\mathfrak{M}, w \models \Box\varphi$ iff $\forall v \in W$ we have wRv implies $\mathfrak{M}, v \models \varphi$.

Kripke semantics

A formula φ is satisfied at a model $\mathfrak{M}$ if there is a point $w \in W$ such that $\mathfrak{M}, w \models \varphi$.

A formula φ is valid in a frame $\mathfrak{F} = (W, R)$ (written: $\mathfrak{F} \models \varphi$) if for any valuation V on $\mathfrak{F}$ and any point $w \in W$ we have $\mathfrak{M}, w \models \varphi$.

Exercise: For what frames do we have:

1. $\mathfrak{F} \models p \to \Diamond p$,
2. $\mathfrak{F} \models \Diamond\Diamond p \to \Diamond p$.

1. $\mathfrak{F} \models p \to \Diamond p$ iff $\mathfrak{F} \models \forall x\, Rxx$,
2. $\mathfrak{F} \models \Diamond\Diamond p \to \Diamond p$ iff $\mathfrak{F} \models \forall x \forall y \forall z (Rxy \wedge Ryz \to Rxz)$.

Kripke semantics

There exists a large class of Sahlqvist formulas such that if a frame validates a Sahlqvist formula it is First-Order definable.

Löb's formula

$$\text{Löb} = \Box(\Box p \to p) \to \Box p$$

Grzegorczyk's formula

$$\text{Grz} = \Box(\Box(p \to \Box p) \to p) \to p$$

are not First-Order definable.

Löb defines transitive coversely well-founded frames and Grz defines transitive Nötherian frames.

Generated subframes

Let (W, R) be a Kripke frame. Let $W' \subseteq W$ and $R' = (W' \times W') \cap R$. Then (W', R') is called a subframe of (W, R).

A subframe is a generated subframe if

$$w \in W' \text{ and } wRv \text{ imply } v \in W'.$$

Proposition. Let (W', R') be a generated subframe of (W, R). Then for each modal formula φ we have

$$(W, R) \models \varphi \text{ implies } (W', R') \models \varphi.$$

p-morphisms

Let (W, R) and (W', R') be Kripke frames and $f : W \to W'$ a map.

f is called a p-morphism if

1. wRv implies $f(w)Rf(v)$,
2. $f(w)Rv$ implies $\exists u \in W$ such that wRu and $f(u) = v$.

If f is an onto p-morphism, then (W', R') is called a p-morphic image of (W, R).

Proposition. Let (W', R') be a p-morphic image of (W, R). Then for each modal formula φ we have

$$(W, R) \models \varphi \text{ implies } (W', R') \models \varphi.$$

Disjoint unions

Let $\mathcal{A} = \{(W_i, R_i) : i \in I\}$ be a family of Kripke frames. The disjoint union $\biguplus_{i \in I}(W_i, R_i)$ of $\mathcal{A}$ is the frame (W, R) such that $W = \biguplus_{i \in I} W_i$ and $R = \bigcup_{i \in I} R_i$.

Proposition. $\mathcal{A} = \{(W_i, R_i) : i \in I\}$ be a family of Kripke frames. Then for each modal formula φ we have

$$(W_i, R_i) \models \varphi \text{ iff } \biguplus_{i \in I}(W_i, R_i) \models \varphi.$$

Bisimulations

Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two Kripke models. We say that a nonempty relation $B \subseteq W \times W'$ is a bisimulation if wBw' implies:

1. $\mathfrak{M}, w \models p$ iff $\mathfrak{M}', w' \models p$ for each $p \in \mathsf{Prop}$,
2. wRv implies $\exists v' \in W'$ such that $w'R'v'$ and vBv', (forth condition)
3. $w'Rv'$ implies $\exists v \in W$ such that wRv and vBv', (back condition)

If w and w' are linked by a bisimulation, they are called bisimilar. Two models are bisimilar if there exists a bisimulation between them.

Theorem. Bisimilar points satisfy the same modal formulas.

Bisimulations

Two points may satisfy the same modal formulas but not be bisimilar.

A Kripke model $\mathfrak{M} = (W, R, V)$ is called image finite if for every point $w \in W$ the set $R(w)$ is finite, where

$$R(w) = \{v \in W : wRv\}$$

Theorem (Hennessy-Milner) On image finite models if two points satisfy the same modal formulas, then they are bisimilar.

Theorem (van Benthem) On (finite) Kripke models modal logic is a bisimulation invariant-fragment of First-Order logic.

Normal modal logics

A normal modal logic is a set of formulas that contains the K-axioms

$$\Box(p \to q) \to (\Box p \to \Box q)$$

and is closed under the rules of Modus Ponens (MP), Necessitation (N) and Uniform Substitution (US)

$\frac{\varphi, \varphi \to \psi}{\psi}$ (MP)

$\frac{\varphi}{\Box \varphi}$ (N)

$\frac{\varphi(p_1, \ldots, p_n)}{\varphi(\psi_1, \ldots, \psi_n)}$ (N)

The least normal modal logic is called the basic modal logic and is denoted by K.

Normal modal logics

Let L be a normal modal logic. The least normal modal logic containing L and a formula φ is denoted by $L + \varphi$.

- $\mathsf{KT} = \mathsf{K} + (p \to \Diamond p)$,
- $\mathsf{K4} = \mathsf{K} + (\Diamond\Diamond p \to \Diamond p)$,
- $\mathsf{S4} = \mathsf{K4} + (p \to \Diamond p)$.

We often write $\vdash_L \varphi$ to mean $\varphi \in L$.

Soundness and Completeness

A modal logic L is said to be sound and complete with respect to a class $\mathcal{C}$ of frames if for every formula φ we have:

$$\vdash_L \varphi \text{ iff } \mathcal{C} \models \varphi$$

where $\mathcal{C} \models \varphi$ if $\mathfrak{F} \models \varphi$ for every $\mathfrak{F} \in \mathcal{C}$.

It is well know that

- KT is sound and complete with respect to the class of reflexive frames.
- K4 is sound and complete with respect to the class of transitive frames.
- S4 is sound and complete with respect to the class of reflexive and transitive frames.

Sahlqvist Theorem. Every logic axiomatized by Sahlqvist formulas is Kripke complete wrt first-order definable frames.

The finite model property

A modal logic L is said to have the finite model property (FMP) if there is a class $\mathcal{C}$ of finite frames such that for every formula φ we have:

$$\vdash_L \varphi \text{ iff } \mathcal{C} \models \varphi$$

KT, K4 and S4 have the finite model property.

Part 2: Topological semantics

Topological semantics

- Topological semantics of modal logic extends relational semantics.
- It provides a richer semantic landscape by bringing in a geometric/spatial interpretations.
- There are important Kripke incomplete logics (e.g., the provability logic **GLP**) which are topologically complete.
- Topological models provide (interesting) semantics for knowledge and belief.

Topological spaces

A topological space is a pair (X, τ), where X is a set and $\tau \subseteq \mathcal{P}(X)$ such that

- $X, \varnothing \in \tau$,
- τ is closed under finite intersections,
- τ is closed under arbitrary unions.

Elements of τ are called open sets.

Complements of open sets are called closed sets.

An open set containing $x \in X$ is called an open neighbourhood of x.

Topological spaces

The interior of a set $A \subseteq X$ is the largest open set contained in A and is denoted by $\mathrm{Int}(A)$.

That is, $\mathrm{Int}(A) = \bigcup\{U \in \tau : U \subseteq A\}$

The closure of A is the least closed set containing A and is denoted by $\mathrm{Cl}(A)$.

That is, $\mathrm{Cl}(A) = \bigcap\{F : X \setminus F \in \tau, A \subseteq F\}$.

It is easy to check that $\mathrm{Cl}(A) = X \setminus \mathrm{Int}(X \setminus A)$.

Alexandroff spaces

Any reflexive and transitive Kripke frame (X, R) (pre-order) can be seen as a topological space (X, τ_R), where τ_R consists of all up-sets of (X, R).

A set $U \subseteq X$ is an up-set if $x \in U$ and xRy imply $y \in U$.

A topological space (X, τ) is called an Alexandroff space if $\tau = \tau_R$ for some reflexive and transitive order R on X.

Theorem. The following are equivalent.

- (X, τ) is Alexandroff,
- τ is closed under arbitrary intersections,
- Every point $x \in X$ has a least open neighbourhood (the intersection of all its open neighbourhoods).

Alexandroff spaces

Let (X, R) be a reflexive and transitive Kripke frame. For each $x \in X$, $R(x) = \{y \in X : xRy\}$ is the least open neighbourhood.

The real line is an example of a non-Alexandroff topological space.

Semantics

Let $\mathfrak{M} = (X, R, V)$ be a reflexive and transitive Kripke model.

Let
$$[\![\varphi]\!] = \{x \in X : \mathfrak{M}, w \models \varphi\}.$$

Then

$$[\![\Box\varphi]\!] = \{x \in X : R(x) \subseteq [\![\varphi]\!]\}$$
$$[\![\Diamond\varphi]\!] = \{x \in X : R(x) \cap [\![\varphi]\!] \neq \varnothing\}$$

$$[\![\Box\varphi]\!] = \mathrm{Int}_{\tau_R}([\![\varphi]\!])$$
$$[\![\Diamond\varphi]\!] = \mathrm{Cl}_{\tau_R}([\![\varphi]\!])$$

Topological semantics

A topological model $\mathcal{M} = (X, \tau, \nu)$ is a tuple where

(X, τ) is a topological space and ν a valuation, i.e., a map $\nu : \mathsf{Prop} \to \mathcal{P}(X)$.

The semantics for modal formulas is defined by the following inductive definition, where p is a propositional variable:

$$
\begin{array}{ll}
[\![\bot]\!] = \varnothing, & [\![p]\!] = \nu(p) \\
[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] & [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!] & [\![\Box\varphi]\!] = \mathrm{Int}[\![\varphi]\!]
\end{array}
$$

Since $\Diamond\varphi = \neg\Box\neg\varphi$, we have $[\![\Diamond\varphi]\!] = \mathrm{Cl}[\![\varphi]\!]$.

Topological semantics

A pointwise definition of the topological semantics is as follows:

$\mathcal{M}, x \models \Box\varphi$ iff $\exists U \in \tau$ such that $x \in U$ and $\forall y \in U$ $\mathcal{M}, y \models \varphi$.

$\mathcal{M}, x \models \Diamond\varphi$ iff $\forall U \in \tau$ such that $x \in U$, $\exists y \in U$ with $\mathcal{M}, y \models \varphi$.

A formula φ is valid in a space X (written: $X \models \varphi$) if for any valuation ν on X and any point $x \in X$ we have $(X, \nu), x \models \varphi$.

Example: Spoon

Topological semantics

Let $\mathfrak{M} = (X, R, V)$ where R is reflexive and transitive. Let $\tau_R =$ the set of upsets of $(X, R)\}$. Let $\mathcal{M} = (X, \tau_R, V)$

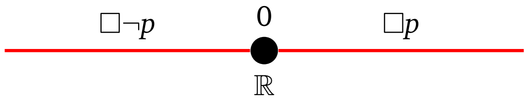
Lemma. For every formula φ and $x \in X$ we have

$$\mathfrak{M}, x \models \varphi \;\text{ iff }\; \mathcal{M}, x \models \varphi$$

Proof: Exercise.

Topological semantics

Let $\nu(p) = (0, \infty)$.



$$\nu(\Box p \vee \Box\neg p) \neq \mathbb{R}$$

Next lecture: Topological completeness

Theorem (McKinsey and Tarski, 1944)

1. S4 is complete wrt all topological spaces.
2. S4 is complete wrt any dense-in-itself metrizable space X.
3. S4 is complete wrt the real line $\mathbb{R}$.
4. S4 is complete wrt the rational line $\mathbb{Q}$.

Recap

- Relational semantics of modal logic,
- Topological semantics.

Topo-bisimulation

Definition. A topological bisimulation or simply a topo-bisimulation between two topo-models $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M} = (X', \tau', \nu')$ is a non-empty relation $T \subseteq X \times X'$ such that if xTx' then:

1. $x \in \nu(p) \Leftrightarrow x' \in \nu'(p)$ for each $p \in \mathsf{Prop}$.
2. (forth): $x \in U \in \tau \Rightarrow \exists U' \in \tau'$ such that $x' \in U'$ and $\forall y' \in U'$ $\exists y \in U$ such that yTy'.
3. (back): : $x' \in U' \in \tau' \Rightarrow \exists U \in \tau$ such that $x \in U$ and $\forall y \in U$ $\exists y' \in U'$ such that yTy'.

As in the relational case if two points are linked by a topo-bisimulation, they are called topo-bisimilar.

Topo-bisimulation

Theorem. Let $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M}' = (X', \tau', \nu')$ be two topo-models. Let $x \in X$ and $x' \in X'$ be topo-bisimilar points. Then for each modal formula φ we have

$$\mathcal{M}, x \models \varphi \text{ iff } \mathcal{M}', x' \models \varphi$$

That is, modal formulas are invariant under topo-bisimulations.

Proof sketch: By induction on the complexity of φ. The only interesting case is $\varphi = \Diamond\psi$.

Topo-bisimulation

Theorem. Let $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M}' = (X', \tau', \nu')$ be two finite topo-models. Let $x \in X$ and $x' \in X'$ be points satisfying the same modal formulas. Then there is a topo-bisimulation connecting x and x'. Finite modally equivalent models are topo-bisimilar.

Open question: What is the analogue of image finiteness for topo-models?

Ten Cate, Gabelaia and Sutretov (2009) proved an analogue of van Benthem characterization theorem for topo-bisimulation on topo-models.

Open subspaces

Let X be a topological space and $Y \subseteq X$.

We can define a subspace topology on Y by taking $\{U \cap Y : U \text{ is an open set in } X\}$.

Let Y be an open subset of X.

Proposition. Let X be a topological space and $Y \subseteq X$ an open subset. Then for each modal formula φ we have

$$X \models \varphi \text{ implies } Y \models \varphi.$$

Interior maps

Let X and Y be topological spaces and $f : X \to Y$ a map.

f is called a continuous if for each open set $U \subseteq Y$ the set $f^{-1}(U)$ is open in X.

f is called a open if for each open set $V \subseteq X$ the set $f[V]$ is open in Y.

f is called interior if it is both continuous and open.

Exercise: Characterize interior maps between Alexandroff spaces?

Proposition. Let X and Y be topological spaces and $f : X \to Y$ an onto interior map. Then for each modal formula φ we have

$$X \models \varphi \text{ implies } Y \models \varphi.$$

Topological sums

Let $\mathcal{A} = \{X_i : i \in I\}$ be a family of topological spaces. The topological sum $\bigoplus_{i \in I} X_i$ of $\mathcal{A}$ is the space X such that $X = \biguplus_{i \in I} X_i$ and $U \subseteq X$ is open iff $U \cap X_i$ is open for each $i \in I$.

Proposition. $\mathcal{A} = \{X_i : i \in I\}$ be a family of topological spaces. Then for each modal formula φ we have

$$\text{For each } i \in I, X_i \models \varphi \text{ iff } \bigoplus_{i \in I} X_i \models \varphi.$$

Part 3: Topological completeness

Topological soundness and completeness

A modal logic L is said to be sound and complete with respect to a class $\mathcal{C}$ of topological spaces if for every formula φ we have:

$$\vdash_L \varphi \text{ iff } \mathcal{C} \models \varphi$$

where $\mathcal{C} \models \varphi$ if $X \models \varphi$ for every $X \in \mathcal{C}$.

Recall that $\mathsf{S4} = \mathsf{K} + (p \to \Diamond p) + (\Diamond\Diamond p \to \Diamond p)$.

Theorem. S4 is topologically complete. In fact, any Kripke complete modal logic $L \supseteq \mathsf{S4}$ is topologically complete.

That S4 is topologically complete also follows from the topo-canonical model construction.

Topological soundness

It is well known that S4 can be axiomatized by

1. $\Diamond\bot \leftrightarrow \bot$,
2. $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$,
3. $p \rightarrow \Diamond p$,
4. $\Diamond\Diamond p \rightarrow \Diamond p$.

Or alternatively by

1. $\Box\top \leftrightarrow \top$,
2. $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$,
3. $\Box p \rightarrow p$,
4. $\Box p \rightarrow \Box\Box p$.

Kuratowski's axioms and S4

Kuratowski's axioms closely resemble the axioms of S4:

$\mathsf{Cl}(\varnothing) = \varnothing$	$\Diamond\bot \leftrightarrow \bot$
$\mathsf{Cl}(A \cup B) = \mathsf{Cl}(A) \cup \mathsf{Cl}(B)$	$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$
$A \subseteq \mathsf{Cl}(A)$	$p \to \Diamond p$
$\mathsf{Cl}(\mathsf{Cl}(A)) \subseteq \mathsf{Cl}(A)$	$\Diamond\Diamond p \to \Diamond p$

$\mathsf{Int}(X) = X$	$\Box\top \leftrightarrow \top$
$\mathsf{Int}(A \cap B) = \mathsf{Int}(A) \cap \mathsf{Int}(B)$	$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$
$\mathsf{Int}(A) \subseteq A$	$\Box p \to p$
$\mathsf{Int}(A) \subseteq \mathsf{Int}(\mathsf{Int}(A))$	$\Box p \to \Box\Box p$

This entails soundness of S4 for topological spaces.

McKinsey-Tarski Theorem

A topological space X is called dense-in-itself if X has no isolated points, i.e., there is no point $x \in X$ such that $\{x\}$ is open.

Theorem

(McKinsey-Tarski, 1944) S4 is the logic of an arbitrary (nonempty) dense-in-itself metric space.

Remark

The original McKinsey-Tarski result had an additional assumption that the space is separable. In their 1963 book Rasiowa and Sikorski showed that this additional condition can be dropped. Their proof uses the Axiom of Choice.

How to prove the McKinsey-Tarski Theorem

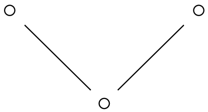
- Since S4 has the finite model property, each non-theorem is refuted on a finite Kripke frame, where the binary relation is reflexive and transitive. Let is call such frames S4-frames.
- Since refuting a formula at a point x of an S4-frame $\mathfrak{F}$ only requires the points from $R[x]$, we may assume that $\mathfrak{F}$ is rooted, meaning that there is a point, called a root, such that every point is seen from it.
- Given a dense-in-itself metric space X, the key is to transfer each such finite refutation to X. This can be done by defining an onto interior map $f : X \to \mathfrak{F}$.

How to prove the McKinsey-Tarski Theorem

- Interior maps satisfy $\mathsf{Int}(f^{-1}(A)) = f^{-1}(\mathsf{Int}(A))$ or equivalently $\mathsf{Cl}(f^{-1}(A)) = f^{-1}(\mathsf{Cl}(A))$.
- Constructing such a map from X onto an arbitrary finite rooted S4-frame is the main challenge in proving the McKinsey-Tarski theorem.
- But as soon as such a map is constructed, the rest of the proof is easy: each non-theorem φ of S4 is refuted on a finite rooted S4-frame $\mathfrak{F}$. Utilizing $f : X \to \mathfrak{F}$, we can pull the refutation of φ from $\mathfrak{F}$ to X. Thus, each non-theorem of S4 is refuted on X, yielding completeness of S4 with respect to X.

Easy example

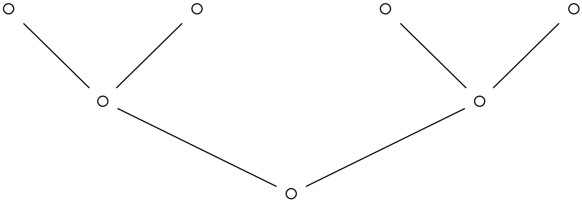
Let X be the real line $\mathbb{R}$ and $\mathfrak{F}$ the two-fork



Define $f : \mathbb{R} \to \mathfrak{F}$ by sending 0 to the root, the negatives to one maximal node, and the positives to the other maximal node.

More complicated example

To define f from $\mathbb{R}$ onto



send the Cantor set to the root. Next send to the children of the root the Cantor sets of the appropriate intervals modulo 2. Send the leftovers to the appropriate maximal nodes.

How to handle clusters

Define f from $\mathbb{R}$ onto the two-point cluster

$$\circ \longleftrightarrow \circ$$

by sending the rationals to one node and the irrationals to the other.

More generally, given an n-cluster, partition $\mathbb{R}$ into n-many dense subsets, and send the equivalence classes to the corresponding nodes in the cluster.

General construction

- Let $\mathfrak{F}$ be a finite tree of clusters. Send the Cantor set to the root cluster (and divide it appropriately, depending on the size of the cluster).
- Depending on the branching of the root cluster, send the Cantor sets of the appropriate intervals modulo the branching. Again, divide them appropriately, depending on the size of the clusters.
- Continue the process as you move up. The end result is your desired map $f : \mathbb{R} \to \mathfrak{F}$.
- Each finite rooted S4-frame is a p-morphic image of an appropriate finite tree of clusters. Combine this with the above to conclude that each finite rooted S4-frame is an image of $\mathbb{R}$.

In summary: Topological completeness

Theorem (McKinsey and Tarski, 1944)

1. S4 is complete wrt all topological spaces.
2. S4 is complete wrt any dense-in-itself metrizable space X.
3. S4 is complete wrt the real line $\mathbb{R}$.
4. S4 is complete wrt the rational line $\mathbb{Q}$.

Recap

Theorem (McKinsey and Tarski)

1. S4 is complete wrt all topological spaces.
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Part 4: Topo-definability

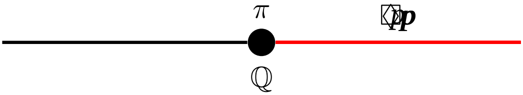
Enriched language

Let us add the universal modality $[\forall]$ to our language.

$$\mathfrak{M}, w \models [\forall]\varphi \text{ iff } \forall v \text{ we have } \mathfrak{M}, v \models \varphi.$$

Let

$$\varphi = [\forall](\Diamond p \leftrightarrow \Box p) \to [\forall]p \vee [\forall]\neg p$$



φ is false in $\mathbb{Q}$

φ is true in $\mathbb{R}$ and $\mathbb{R}^2$

Topo-definability

A class $\mathcal{C}$ of topological spaces is called modally definable iff there exists a set of modal formulas Φ such that for each topological space X we have

$$X \in \mathcal{C} \text{ iff } X \models \Phi.$$

Theorem.

1. Neither compactness nor connectedness is topo-definable.
2. None of the separation axioms T_0, T_1, and T_2 is topo-definable.

Proof sketch. (1) Start with a singleton space and take an infinite topological sum. You obtain an infinite discrete space which is neither compact nor connected. Use the preservation result on topological sums.

(2) Start with $\mathbb{R}$ which satisfies all the separation axioms. Consider a two point trivial space (two point cluster), which is an image via an interior map. Now use the preservation result of onto interior maps.

The logic of discrete spaces

We have

$$\begin{aligned} X \models p \to \Box p \quad & \text{iff} \quad \text{every subset of } X \text{ is open} \\ & \text{iff} \quad X \text{ is discrete.} \end{aligned}$$

Therefore, $p \to \Box p$ (or equivalently $\Diamond p \to p$) topo-defines the class of discrete spaces.

S5-spaces

Recall that

$$\mathsf{S5} = \mathsf{S4} + (p \to \Box\Diamond p) = \mathsf{S4} + (\Diamond p \to \Box\Diamond p).$$

$$X \models \Diamond p \to \Box\Diamond p \quad \text{iff} \quad \mathsf{Cl}(A) \subseteq \mathsf{Int}(\mathsf{Cl}(A)) \text{ for each } A \subseteq X$$
$$\text{iff} \quad \text{every closed subset of } X \text{ is open.}$$

Therefore, S5 topo-defines the class of topological spaces in which every closed subset is open.

In fact, S5 is also complete wrt these spaces. Why?

(Hint: use the fact that S5 is complete wrt Krpke frames with a universal relation ($\forall x \forall y Rxy$)).

S4.2 and extremally disconnected spaces

A topological space X is extremally disconnected if the closure of every open subset of X is open.

$$\mathsf{S4.2} = \mathsf{S4} + (\Diamond\Box p \to \Box\Diamond p).$$

$$\begin{aligned} X \models \Diamond\Box p \to \Box\Diamond p \quad & \text{iff} \quad \mathsf{Cl}(\mathsf{Int}(A)) \subseteq \mathsf{Int}(\mathsf{Cl}(A)) \text{ for each } A \subseteq X \\ & \text{iff} \quad \mathsf{Cl}(\mathsf{Int}(A)) = \mathsf{Int}(\mathsf{Cl}(\mathsf{Int}(A))) \text{ for each } A \subseteq X \\ & \text{iff} \quad \text{the closure of every open subset of } X \text{ is open} \\ & \text{iff} \quad X \text{ is extremally disconnected.} \end{aligned}$$

Therefore, S4.2 topo-defines the class of extremally disconnected spaces.

In fact, S4.2 is also complete wrt extremally disconnected spaces. Why?

(Hint: use the fact that S4.2 is complete wrt rooted ($\exists x \forall y Rxy$), directed ($\forall x \forall y (Rxy \wedge Rxz) \to \exists u (Ryu \wedge Rzu)$), reflexive and transitive frames

S4.Grz and hereditarily irresolvable spaces

Recall that

$$\mathsf{S4.Grz} = \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to \Box p.$$

A space X is resolvable if it can be represented as the union of two disjoint dense subsets.

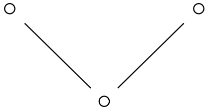
X is irresolvable if it is not resolvable, and it is hereditarily irresolvable if every subspace of X is irresolvable.

Theorem (Esakia, 1981) S4.Grz topo-defines hereditarily irresolvable spaces.

In fact, S4.Grz is also complete wrt hereditarily irresolvable spaces.

The logic of intervals

If we consider the smaller Boolean algebra generated by the open intervals of $\mathbb{R}$, then we can only pick up the two-fork



Theorem (Aiello, van Benthem, G. Bezhanishvili, 2003)

The logic of the two-fork is the logic of the Boolean algebra generated by the open intervals of $\mathbb{R}$.

Part 5: Intuitionistic logic and Gödel translation

Constructive reasoning

One of the cornerstones of classical reasoning is the law of excluded middle $p \vee \neg p$.

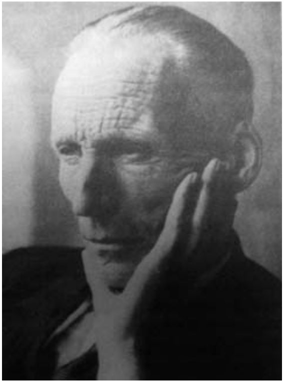
Constructive viewpoint: Truth = Proof.

The law of excluded middle $p \vee \neg p$ is constructively unacceptable.

For example, we do not have a proof of Goldbach's conjecture nor are we able to show that this conjecture does not hold.

Constructive reasoning

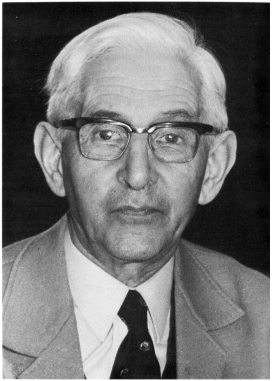
On the grounds that the only accepted reasoning should be constructive, the dutch mathematician L. E. J. Brouwer rejected classical reasoning.



Luitzen Egbertus Jan Brouwer (1881 - 1966)

Intuitionistic logic

In 1930's Brouwer's ideas led his student Heyting to introduce intuitionistic logic which formalizes constructive reasoning.



Arend Heyting (1898 - 1980)

Intuitionistic logic

Roughly speaking, the axiomatization of intuitionistic logic is obtained by dropping the law of excluded middle from the axiomatization of classical logic.

CPC = classical propositional calculus
IPC = intuitionistic propositional calculus.

The law of excluded middle is not derivable in intuitionistic logic. So $\mathsf{IPC} \subsetneq \mathsf{CPC}$.

In fact,

$$\mathsf{CPC} = \mathsf{IPC} + (p \vee \neg p).$$

There are many logics in between IPC and CPC

Superintuitionistic logics

A superintuitionistic logic is a set of formulas containing IPC and closed under the rules of substitution and Modus Ponens.

Superintuitionistic logics contained in CPC are often called intermediate logics because they are situated between IPC and CPC.

As we will see, intermediate logics are exactly the consistent superintuitionistic logics.

Since we are interested in consistent logics, we will mostly concentrate on intermediate logics.

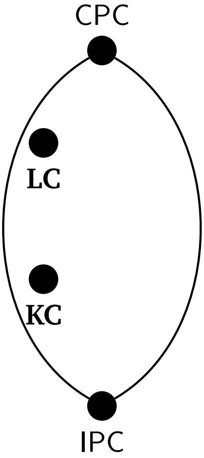
Intermediate logics

$\mathbf{LC} = \mathsf{IPC} + (p \to q) \vee (q \to p)$

Gödel-Dummett calculus

$\mathbf{KC} = \mathsf{IPC} + (\neg p \vee \neg\neg p)$

weak law of excluded middle



Topological semantics of intuitionistic logic

Let IPC denote Intuitionistic Propositional Calculus.

IPC admits topological semantics.

Recall that he language of IPC is defined by the grammar

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi,$$

where $p \in \mathsf{Prop}$

We use the shorthand $\neg\varphi := \varphi \to \bot$.

Topological semantics of intuitionistic logic

Let (X, τ) be a topological space.

An intuitionistic valuation is a map $V : \mathsf{Prop} \to \tau$.

We extend it to all formulas by the following inductive definition:

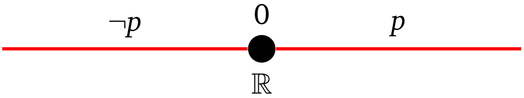
$$
\begin{array}{ll}
[\![\bot]\!] = \varnothing, & [\![p]\!] = V(p) \\
[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] & [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\varphi \to \psi]\!] = \mathsf{Int}((X \setminus [\![\varphi]\!]) \cup [\![\psi]\!]) & [\![\neg\varphi]\!] = \mathsf{Int}(X \setminus [\![\varphi]\!])
\end{array}
$$

Validity is defined in the same way as in the topological semantics of modal logic.

Theorem. IPC is sound and complete wrt topological semantics.

Topological semantics of IPC

Let $V(p) = (0, \infty)$.



$$[\![p \vee \neg p]\!] \neq \mathbb{R}$$

This is a topological refutation of the law of excluded middle.

Gödel translation

There is a celebrated Gödel translation from IPC to S4 defined as follows:

- $Tr(p) = \Box p$,
- $Tr(\varphi \wedge \psi) = Tr(\varphi) \wedge Tr(\psi)$,
- $Tr(\varphi \vee \psi) = Tr(\varphi) \vee Tr(\psi)$,
- $Tr(\varphi \to \psi) = \Box(Tr(\varphi) \to Tr(\psi))$.

This translation is full and faithful, i.e.

$$\mathsf{IPC} \vdash \varphi \text{ iff } \mathsf{S4} \vdash Tr(\varphi)$$

$$\mathsf{IPC} \vdash \varphi \text{ iff } \mathsf{S4.Grz} \vdash Tr(\varphi)$$

S4 is the least modal companion of IPC and S4.Grz is the greatest modal companion of IPC.

Key lemma

Lemma. Let X be a topological space and φ an intuitionistic formula. Then

$$X \models_{\mathsf{IPC}} \varphi \text{ iff } X \models_{\mathsf{S4}} Tr(\varphi).$$

Proof: Exercise.

Exercise: Derive

$$\mathsf{IPC} \vdash \varphi \quad \text{iff} \quad \mathsf{S4} \vdash Tr(\varphi)$$

from the above lemma.

Part 6: Derived set semantics

Derived set operator

Let (X, τ) be a topological space and $Y \subseteq X$. A point $x \in X$ is called a limit point of Y if for any $U \in \tau$ with $x \in U$ we have that $(U \setminus \{x\}) \cap Y \neq \varnothing$.

Let $d(Y)$ denote the set of the limit points of Y.

In general, $Y \nsubseteq d(Y)$

Also $\mathsf{Cl}(Y) = Y \cup d(Y)$ and

- $d(\varnothing) = \varnothing$,
- $d(A \cup B) = d(A) \cup d(B)$,
- $d(d(A)) \subseteq A \cup d(A)$.

Proof: Exercise.

Derived set semantics

Let (X, τ) be a topological space and $V : \mathsf{Prop} \to \mathcal{P}(X)$ a valuation. We defined a derived set semantics of modal logic (d-semantics, for short).

Let $x \in X$, $\mathcal{M} = (X, \tau, \nu)$ and φ be a modal formula.

$$\mathcal{M}, w \models_d p \text{ iff } w \in V(p),$$
$$\mathcal{M}, w \models_d \varphi \vee \psi \text{ iff } \mathcal{M}, w \models_d \varphi \text{ or } \mathcal{M}, w \models_d \psi,$$
$$\mathcal{M}, w \models_d \neg\varphi \text{ iff } \mathcal{M}, w \not\models_d \varphi,$$
$$\mathcal{M}, x \models_d \Diamond\varphi \text{ iff } \forall U \in \tau \ (x \in U \Rightarrow \exists y \in U \setminus \{x\} : \mathcal{M}, y \models_d \varphi).$$
$$\mathcal{M}, x \models_d \Box\varphi \text{ iff } \exists U \in \tau \ (x \in U \Rightarrow \forall y \in U \setminus \{x\}, \mathcal{M}, y \models_d \varphi).$$

In other terms, $[\![\Diamond\varphi]\!] = d[\![\varphi]\!]$.

d-validity is defined as for the closure-interior semantics.

Derived set semantics

Let

$$\mathsf{wK4} = \mathsf{K} + (\Diamond\Diamond p \to p \vee \Diamond p).$$

wK4 is sound and complete wrt weakly transitive frame, i.e., frame satisfying

$$\forall x \forall y \forall z (Rxy \wedge Ryz) \to (Rxz \vee x = z)).$$

Theorem. wK4 is sound wrt all topological spaces.

Split translation

There is a split translation from S4 to wK4 defined as follows:

- $Sp(p) = p$,
- $Sp(\varphi \vee \psi) = Sp(\varphi) \vee Sp(\psi)$,
- $Sp(\neg\varphi) = \neg Sp(\varphi)$,
- $Sp(\Diamond\varphi) = Sp(\varphi) \vee \Diamond Sp(\varphi)$,
- $Sp(\Box\varphi) = Sp(\varphi) \wedge \Box Sp(\varphi)$.

Theorem. For each modal formula φ we have

$$\mathsf{S4} \vdash \varphi \text{ iff } \mathsf{wK4} \vdash Sp(\varphi).$$

Key Lemma. Let X be a topological space and φ a modal formula. Then

$$X \models \varphi \text{ iff } X \models_d Sp(\varphi).$$

Exercise: Derive the above theorem from the key lemma.

Recap

- Topo-definability and undefinability,
- Topological semantics of intuitionistic logic and Gödel translation,
- Derived set semantics and wK4.

Recap

1. $x \in \mathsf{Cl}(A)$ iff $\forall U \in \tau$ $(x \in U \Rightarrow \exists y \in U$ with $y \in A)$.
2. $\mathcal{M}, x \models_{\mathsf{Cl}} \Diamond\varphi$ iff $\forall U \in \tau$ $(x \in U \Rightarrow \exists y \in U$ with $\mathcal{M}, y \models_{\mathsf{Cl}} \varphi)$.
3. $x \in d(A)$ iff $\forall U \in \tau$ $(x \in U \Rightarrow \exists y \in U \setminus \{x\}$ with $y \in A)$.
4. $\mathcal{M}, x \models_d \Diamond\varphi$ iff $\forall U \in \tau$ $(x \in U \Rightarrow \exists y \in U \setminus \{x\}$ with $\mathcal{M}, y \models_d \varphi)$.

Expressive power

In the d-semantics we can express closure as $\mathsf{Cl}(A) = d(A) \cup A$.

That is by the formula $\Diamond p \vee p$.

However, in the Cl-semantics we cannot express the derivative.

For this we need to find topo-models $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M}' = (X', \tau', \nu')$, a topo-bisimulation $T \subseteq X \times X'$, points $x \in X$ and $x' \in X'$ and a formula φ such that xTx' and

$$\mathcal{M}, x \models_d \varphi \text{ and } \mathcal{M}', x' \not\models_d \varphi.$$

Exercise: Show the above.

d-bisimulations

A d-bisimulation between two topo-models $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M} = (X', \tau', \nu')$ is a non-empty relation $T \subseteq X \times X'$ such that if xTx' then:

1. $x \in \nu(p) \Leftrightarrow x' \in \nu'(p)$ for each $p \in \mathsf{Prop}$.
2. (forth): $x \in U \in \tau \Rightarrow \exists U' \in \tau'$ such that $x' \in U'$ and $\forall y' \in U' \setminus \{x'\}\ \exists y \in U \setminus \{x\}$ such that yTy'.
3. (back): : $x' \in U' \in \tau' \Rightarrow \exists U \in \tau$ such that $x \in U$ and $\forall y \in U \setminus \{x\}\ \exists y' \in U' \setminus \{x'\}$ such that yTy'.

Topo-bisimulation

Theorem. Let $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M}' = (X', \tau', \nu')$ be two topo-models. Let $x \in X$ and $x' \in X'$ be d-bisimilar points. Then for each modal formula φ we have

$$\mathcal{M}, x \models_d \varphi \text{ iff } \mathcal{M}', x' \models_d \varphi.$$

Derivative in Alexandroff spaces

Lemma. Let (X, τ) be an Alexandroff space corresponding to a preorder (X, R). Then for $x \in X$ and $A \subseteq X$ we have $x \in d(A)$ iff xRy for some $y \in A$ such that $x \neq y$.

Proof. Let $x \in d(A)$. Then $R(x) \setminus \{x\} \cap A \neq \varnothing$. So there is $y \in A$ such that xRy and $y \neq x$. The other direction is similar.

Thus, $d(A) = R_-^{-1}(A)$, where $R_- = R \setminus \{(x, x) : x \in X\}$ is the irreflexive core of R.

So d is the diamond of the irreflexive core of R.

Thus, the topological Kripke frames are irreflexive weakly transitive ones.

Derivative in Alexandroff spaces

Note that with every weakly transitive frame (X, R) we can associate a topology of all R-upsets.

This topology will be the same as the topology of all upsets of the reflexive closure of R.

Indeed, $A \subseteq X$ is an upset iff it is an upset for the reflexive closure of R.

If the original R was irreflexive, then $d(A) = R^{-1}(A)$, i.e., the diamond of R.

So if wK4 is complete wrt irreflexive weakly transitive frames, it is topologically complete.

Topological completeness of wK4

Theorem (Esakia, 2001) wK4 is sound and complete wrt all topological spaces.

Proof: We have shown soundenss already. For completeness, note that by Sahqvist theorem, wK4 is complete wrt weakly transitive frames. Thus, if wK4 $\nvdash \varphi$, there exists a weakly transtive Kripke frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Replace any reflexive point in $\mathfrak{F}$ with an irreflexive two point cluster. Call the new frame $\mathfrak{G}$. Then $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$ (obtained by identifying these two point clusters). So we found an irreflexive weakly transitive frame refuting φ. Thus, wK4 is complete for irreflexive weakly transitive frames. So wK4 is topologically complete.

K4 and T_D-spaces

Definition. A topological space X is said to satisfy the T_D-separation axiom or is simply T_D if for every point $x \in X$ there exist an open U and closed F such that $U \cap F = \{x\}$.

Recall that a topological space X is T_0 is for every $x, y \in X$ such that $x \neq y$, there is an open set U such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

Proposition. Every T_D-space is T_0.

Proof: Let X be a T_D-space and $x, y \in X$ with $x \neq y$. Then $\{x\} = U \cap F$. Then $y \notin U$ or $y \notin F$. So $x \in U$ and $y \notin U$ or $y \in X \setminus F$ and $x \in F$.

Example of a T_0-space that is not T_D. Note if X is an Alexandroff space, then X is T_0 iff X is T_D.

K4 and T_D-spaces

Theorem. A space X is T_D iff $dd(A) \subseteq d(A)$ for every $A \subseteq X$.

Proof: A set A is closed iff $\mathsf{Cl}(A) \subseteq A$ iff $A \cup d(A) \subseteq A$ iff $d(A) \subseteq A$. So $dd(A) \subseteq d(A)$ implies that $d(A)$ is closed for every $A \subseteq X$.

Note that for every $x \in X$ we always have $x \notin d(x)$. By the above, $d(x)$ is closed and so $X \setminus d(x)$ is open. So $\{x\} = \mathsf{Cl}(x) \cap (X \setminus d(x))$, and X is T_D.

Thanks to Yunsong for the correction!

Conversely, suppose $x \notin d(A)$. Then there is an open neighbourhood U of x such that $U \setminus \{x\} \cap A = \varnothing$. By T_D there are open V and closed F such that $\{x\} = V \cap F$. Then $U \cap V$ is an open neighbourhood of x. We show that $(U \cap V) \cap d(A) = \varnothing$. Assume there is $y \in d(A)$ such that $y \in U \cap V$. Then $y \notin F$. So $(U \cap V) \setminus F$ is an open neighbourhood of y that has empty intersection with A, which is a contradiction. So $x \notin \mathsf{Cl}(d(A))$. As $d(A) \subseteq \mathsf{Cl}(A)$, we obtain that $x \notin dd(A)$.

K4 and T_D-spaces

Corollary. A space X is T_D iff $X \models_d \Diamond\Diamond p \to \Diamond p$.

Theorem. K4 is sound and complete with respect to T_D-spaces.

Proof: Soundness has just been proved. For completeness, suppose K4 $\nvdash \varphi$. By Sahlqvist theorem there exists a transitive frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Let $\mathfrak{G} = (\mathbb{N}, <)$. Consider $\mathfrak{F} \times \mathfrak{G}$. Then $\mathfrak{F} \times \mathfrak{G}$ is a transitive irreflexive frame and $\mathfrak{F}$ is its p-morphic image. So $\mathfrak{F} \times \mathfrak{G} \not\models \varphi$. Thus, K4 is complete for irreflexive transitive frames and hence it is topologically complete for T_D-spaces.

GL and scattered spaces

Recall that

$$\mathsf{GL} = \mathsf{K} + (\Box(\Box p \to p) \to \Box p) = \mathsf{K4} + (\Box(\Box p \to p) \to \Box p).$$

Consider the following Löb's rule: $\Box p \to p / p$. It is known that

$$\mathsf{GL} = \mathsf{K} + \Box p \to p / p.$$

Then X validates this rule if it validates $\Diamond p \vee \neg p / \neg p$, which is equivalent to $\neg(p \wedge \neg\Diamond p) / \neg p$.

By contraposition we obtain that X validates the rule if for each $U \subseteq X$

$$U \neq \varnothing \Rightarrow U \setminus d(U) \neq \varnothing.$$

Not that in the Kripke semantics this means that every subset has a maximal element (i.e., converse well foundedness).

GL and scattered spaces

What does it mean topologically?

Definition. A topological space X is called scattered if every non-empty subset of X has an isolated point.

For each set U every $x \in U$ is either a limit point or an isolated point.

Theorem (Esakia, Simmons). Let X be a topological space. Then

$$X \models_d \text{Löb iff } X \text{ is scattered.}$$

Proof: follows from the above.

GL and scattered spaces

Does an analogue of the McKinsey-Tarski theorem hold for GL?

To some degree yes!

Consider the ordinal space ω^ω with an interval topology. It is known that this is a scattered space. In fact, this is a classic example of a scattered space.

Theorem (Abashidze, Blass). GL is sound an complete for ω^ω.

Proof idea: It is known that GL is complete wrt finite roooted irreflexive transitive trees. We can show that every such tree of height *n* is a "*d*-morphic image" of $\omega^n + 1$. Then every finite tree is a '*d*-morphic image" of ω^ω. Like for M-T this finishes the proof.

Recap

Recap

- Expressivity
- Irreflexive weakly transitive frames are topological
- T_D-spaces
- Topological completeness of wK4, K4 and GL

Separation axioms in d-semantcis

Theorem (G. Bezhanishvili, Esakia, Gabelaia, 2005) K4 is sound and complete wrt Stone spaces (which are normal (i.e., T_4).

Thus, there is no extra formula that is valid on $T_1, \ldots, T_4$ spaces other than $\Diamond\Diamond p \to \Diamond p$.

That is the classes of $T_1, \ldots, T_4$ spaces are not modally definable in d-semantics.

If we enrich the modal language with a difference modality, then T_0 and T_1 spaces are definable (Gabelaia, 2005).

The d-logic of the real line was fully axiomatized by Shehtman (1999). Later a simplified proof was given by Lucero-Bryan (2013).

Logic of T_0-spaces

Is the class of T_0-spaces modally defined in the d-semantics?

Let

$$\mathrm{t}_0 = p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q).$$

Theorem (G. Bezhanishvili, Esakia, Gabelaia, 2011) Let X be a topological space. Then

$$X \models \mathrm{t}_0 \text{ iff } A \cap d(B \cap d(A)) \subseteq d(A) \cup d(B \cap d(B)) \text{ iff } X \text{ is } T_0.$$

A weakly transitive frame (W, R) validates t_0 iff $wRvRw$ implies that wRw or vRv. Such frames are called weakly reflexive.

If we define the cluster of $w \in W$ to be the set of points v so that $vRwRv$ then a weakly transitive frame (W, R) validates t_0 iff every cluster has at most one irreflexive point.

Logic of T_0-spaces

Let

$$\mathsf{wK4T_0} := \mathsf{wK4} + p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q).$$

Theorem. $\mathsf{wK4T_0}$ is sound and complete wrt T_0-spaces.

Split translation

There is a split translation from S4 to wK4 defined as follows:

- $Sp(p) = p$,
- $Sp(\varphi \vee \psi) = Sp(\varphi) \vee Sp(\psi)$,
- $Sp(\neg\varphi) = \neg Sp(\varphi)$,
- $Sp(\Diamond\varphi) = Sp(\varphi) \vee \Diamond Sp(\varphi)$,
- $Sp(\Box\varphi) = Sp(\varphi) \wedge \Box Sp(\varphi)$.

Theorem. For each modal formula φ we have

$$\mathsf{S4} \vdash \varphi \text{ iff } \mathsf{wK4} \vdash Sp(\varphi).$$

Key Lemma. Let X be a topological space and φ a modal formula. Then

$$X \models \varphi \text{ iff } X \models_d Sp(\varphi).$$

Split translation GL and S4.Grz

Theorem. For each modal formula φ we have

$$\mathsf{S4.Grz} \vdash \varphi \text{ iff } \mathsf{GL} \vdash Sp(\varphi).$$

Key Lemma. Let $\mathfrak{F} = (W, R)$ be a weakly transitive frame and $\overline{\mathfrak{F}} = (W, \overline{R})$ its reflexive closure. Then

$$\overline{\mathfrak{F}} \models \varphi \text{ iff } \mathfrak{F} \models Sp(\varphi).$$

Proof of the theorem: $\mathsf{GL} \nvdash Sp(\varphi)$ iff there is a conversely well-founded transitive frame $\mathfrak{F} = (W, R)$ such that $\mathfrak{F} \not\models Sp(\varphi)$ iff $\overline{\mathfrak{F}} \not\models \varphi$ iff $\mathsf{S4.Grz} \not\models \varphi$. Here we used the fact that the reflexivization of a conversely well founded frame is Nötherian.

Part 7: Evidence-based epistemic semantics

Epistemic interpretation

Definition. A topological evidence model or topo-e-model is a tuple (X, τ, E_0, V), where (X, τ) is a topological space, E_0 is a subbasis for τ and V is a valuation.

A subbasis is a set of sets such that every open set is a union of finite intersections of the elements of the subbasis.

The elements $e \in E_0$ represent the basic pieces of evidence the agent has.

A combined evidence or an argument is an evidence the agent can piece together from her basic evidence, i.e. a nonempty finite intersection $e_0 \cap \cdots \cap e_n$ of pieces of basic evidence.

Epistemic interpretation

We say an agent has a basic piece of evidence for a proposition P at world x whenever there exists some $e \in E_0$ such that $x \in e \subseteq P$.

We say that an agent has evidence for P at x if she has a factive argument for P, i.e. if there is a combined evidence $e_0 \cap \cdots \cap e_n$ such that $x \in e_0 \cap \cdots \cap e_n \subseteq P$.

Since the combined evidence constitutes a topological basis, this is exactly the topological interior of P.

Epistemic interpretation

A set $U \subseteq X$ is dense if $\mathsf{Cl}(U) = X$.

I.e., U is dense iff it has a nonempty intersection with every nonempty open set in the topology.

Formally, a justification is simply a dense argument. It cannot be contradicted by any other argument.

We say that our agent believes P at x if she has a justification for P, i.e., if there exists a dense piece of evidence $U \subseteq P$.

We say that the agent knows P at x if she has a factive justification for P, i.e. if $x \in U \subseteq P$.

- $BP = X$ if $\mathrm{Int}(P)$ is dense and nonempty, $\emptyset$ otherwise;
- $KP = \mathrm{Int}(P)$ if $\mathrm{Int}(P)$ is dense, $\emptyset$ otherwise.

Epistemic interpretation

Given any topological space (X, τ), we take
$\tau' = \{U \in \tau : \mathsf{Cl}(U) = X\} \cup \{\varnothing\}$.

(X, τ') is an extremally disconnected space.

Then knowledge is interior with respect to this topology.

Epistemic interpretation

Theorem (A. Baltag, N.B., S. Fernandez Gonzalez, 2019). The logic of any dense-in-itself metric space with the dense interior semantics is S4.2.

Strategy: S4.2 $\nvdash \varphi$ implies there is a finite rooted S4.2-frame $\mathfrak{F} = (W, R)$. Construct a dense-interior map $f : X \to W$. Then $X \not\models \varphi$ in the dense-interior semantics.

For more details:

- Saul Fernandez, Generic models for topological evidence logics, Master's thesis, ILLC, 2018.
- Aybüke Özgün, Evidence in Epistemic Logic: A Topological Perspective. PhD thesis, ILLC, 2017.

Part 8: Connection with mathematics via dimension and set-theoretic priniples

Modal dimension

Definition (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2017)
Modal dimension for topological spaces is defined recursively as follows:

$$
\begin{array}{lll}
\mathrm{mdim}(X) = -1 & \text{if} & X = \varnothing, \\
\mathrm{mdim}(X) \leq n & \text{if} & \mathrm{mdim}(D) \leq n-1 \text{ for } D \text{ nowhere dense in } X, \\
\mathrm{mdim}(X) = n & \text{if} & \mathrm{mdim}(X) \leq n \text{ and } \mathrm{mdim}(X) \not\leq n-1, \\
\mathrm{mdim}(X) = \infty & \text{if} & \mathrm{mdim}(X) \not\leq n \text{ for any } n = -1, 0, 1, 2, \ldots.
\end{array}
$$

Modal dimension

For $n \geqslant 0$, consider the formulas:

$$
\begin{array}{lcl}
\mathsf{bd}_0 & = & \bot, \\
\mathsf{bd}_{n+1} & = & \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \rightarrow p_{n+1}.
\end{array}
$$

Modal dimension

Let $\mathfrak{F}_n$ be the n-element chain.

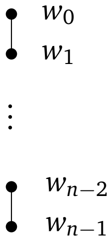


Figure: The n-element chain.

Modal dimension

Theorem (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2017). Let X be a topological space and $n \geqslant 1$. The following are equivalent:

1. $\mathrm{mdim}(X) \leq n - 1$.
2. $X \vDash \mathsf{bd}_n$.
3. $\mathfrak{F}_{n+1}$ is not a continuous and open image of X.

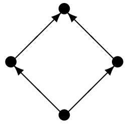
Fact. $\mathrm{mdim}(\mathbb{Q}) = \mathrm{mdim}(\mathbb{R}) = \mathrm{mdim}(\mathbb{R}^n) = \infty$.

Connection to Set Theory

Unexpectedly, some of topological completeness results turned out to be related to set-theoretic statements not provable in ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice).

Definition

Let the diamond be the following **S4**-frame:



Existence of measurable cardinals

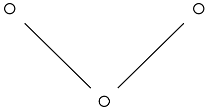
Theorem (with N. Bezhanishvili, J. Lucero-Bryan, J. van Mill, 2020)

There exists a normal space Z whose logic is the logic of the diamond iff there exists a measurable cardinal.

- It remains open whether the requirement that Z be normal can be weakened.
- More generally, the precise connection between topological completeness and the existence of large cardinals is not understood yet.
- Much remains to be done in this direction.

The logic of intervals

If we consider the smaller Boolean algebra generated by the open intervals of $\mathbb{R}$, then we can only pick up the two-fork



Theorem (Aiello, van Benthem, G. Bezhanishvili, 2003)

The logic of the two-fork is the logic of the Boolean algebra generated by the open intervals of $\mathbb{R}$.

Euclidean hierarchy

McKinsey and Tarski theorem implies that modal logic of each Euclidean space is S4.

However, we can distinguish the logics of Euclidean spaces of different dimensions by restricting the valuation to special subsets.

Theorem (van Benthem, G. Bezhanishvili, Gehrke, 2003)

More generally, there is a decreasing sequence of logics $\mathbf{L}_n$ $(n \geqslant 1)$ such that each $\mathbf{L}_n$ is the logic of the Boolean algebra generated by the open hypercubes in $\mathbb{R}^n$. Each $\mathbf{L}_n$ is the logic of the n-product of the two-fork.

This is the beginning of a new story...

Part 9: Polyhedral modal logic

For this part I will use slides from a different presentation.

Topics that we have not covered

- Strong completeness (Kremer, Goldblatt and Hodkinson)
- Multi-agent setting (van Benthem, Baltag, Ožgün)
- Adding fixed-point operators to the modal language (Goldblatt and Hodkinson, Fernandez-Duque, Baltag, N.B.)
- Dynamic topological logics (Fernandez Duque, Björndahl)

Summary

- We discussed in depth topological semantics of modal logic.
- Closure-interior semantics (topo-bisimulations, interior maps, open subspaces, sums, expressivity, definability, completeness, McKinsey-Tarski theorem).
- Derived set semantics, expressivity, definability, completeness, translations.
- Overviewed some related recent research directions.

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Thank you!