Introduction to Topology in and via Logic
Lecture 4

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Plan for the Day

- Announcements
- Introduction to Filters and Filter Convergence.
- Hausdorff Spaces.
- Weaker Separation Axioms.
- Stronger Separation Axioms.

Recap

In the previous three lectures we learned about the basic structures of topology (topological spaces), and the maps preserving this structure (continuous maps).

We have in particular learned about homeomorphisms, which are the correct notion of "structurally identical objects". Our next lectures will be dedicated to studying topological properties, that is, those which distinguish specific topological spaces, and are invariant under homeomorphism.

Epistemic Theories

We start with a topological space X, understood as an epistemic structure. We call $T \subseteq \mathcal{P}(X)$ an *epistemic theory*.

Idea: we want to use theories to decide where we are situated in the epistemic structure. But not all theories are good: $\{X\}$ could support us being anywhere.

Two Possibilities for Good Theories:

- A theory T can be saturated: for each $U \subseteq X$, there is some $S \in T$ such that either $S \subseteq U$ or $S \cap U = \varnothing$;
- A theory T can be definitive: for each $U \in \mathcal{N}(x)$, there is some $V \in T$ such that $V \subseteq U$.

QUESTION: Is there any plausible relationship between these?

It seems reasonable that in a model where we allow agents to know their epistemic position (in the limit) these should be the same thing.

Filters and Filter Bases

Definition

Let X be a set. We say that a collection of subsets $F \subseteq \mathcal{P}X - \{\emptyset\}$ is a *filter base* if it satisfies the following:

- $X \in F$;
- If $A, B \in F$ then $A \cap B \in F$.

We say that a given filter base is a *filter* if it is upwards closed: whenever $A \in F$ and $A \subseteq B$ then $B \in F$.

Example

Example using set $\{a, b, c, d\}$. See blackboard.

Proposition

Let (X, τ) be a topological space, and $x \in X$. Then $\mathcal{N}(x)$ is a filter.

Proof: See Blackboard.

Filter Convergence

Example

Motivating example on reals and Cantor space, see blackboard.

Definition

Let (X, τ) be a topological space and $F \subseteq \tau$ a filter base. We say that the filter base F *converges to a point* x, and that x is the *limit of the filter base*, if and only if for every $U \in \mathcal{N}(x)$, there is some $V \in F$ such that $V \subseteq U$.

Definition

Let (X, τ) be a topological space. We say that X is *Hausdorff*, or T_2, if whenever $x, y \in X$ and $x \neq y$, there there exist two open neighbourhoods $x \in U_x$ and $y \in V_y$ and

$$U_x \cap V_y = \varnothing.$$

Example

Any set with the discrete topology. All but one set with the *indiscrete* topology are *not* Hausdorff.

The example above of the reals shows that the Cantor space is Hausdorff; the Baire space, and the real line are also Hausdorff.

Hausdorff and Filter Convergence

Theorem

Let (X, τ) be a topological space. Then the following are equivalent:

1. *X is Hausdorff;*
2. *For each filter base F, F converges to at most one point;*

Proof: See Blackboard

A note about constructions: products, coproducts and subspaces preserve Hausdorff; quotients in general do not.

Break: $\pm$ 10 minutes.

Weaker Separation Axioms: Frechet

Definition

Let (X, τ) be a topological space. We say that X is *Frèchet* or T_1 if for all $x \neq y \in X$ there exists an open neighbourhood U_x such that $x \in U_x$ and $y \notin U_x$.

Example

Cofinite topology on a set; special case: Zariski topology on $\mathbb{R}$. See blackboard.

Weaker Separation Axioms: Kolmogorov

Definition

Let (X, τ) be a topological space. We say that two points x, y are *topologically distinguishable* if there exists an open neighbourhood $U_{x,y}$ such that either $x \in U_{x,y}$ and $y \notin U_{x,y}$ or $y \in U_{x,y}$ and $x \notin U_{x,y}$. We say that the space X is T_0 if all pairs of points are topologically distinguishable.

Example

Consider $\mathbb{F} = (W, R)$ a Kripke frame. Then one can prove the following two facts, related to these weak separation properties:

1. The topological space induced by $\mathbb{F}$ is T_1 is and only if R corresponds to the identity.
2. The topological space induced by $\mathbb{F}$ is T_0 if and only if R is a partial order (i.e., it is antisymmetric).

In a sense, this means that *only T_0 spaces are interesting for epistemic settings where we want to represent different degrees of knowledge explicitly.*

Stronger Separation axioms: Normal

Definition

Let (X, τ) be a Hausdorff topological space. We say that X is *normal* or T_4 if whenever E, F are disjoint closed sets, then there exist open sets U, V, $E \subseteq U$ and $F \subseteq V$, such that $U \cap V = \varnothing$.

Example

The real line is normal (this is worked out in the notes).

Definition

Let X be a topological space. We say that two disjoint closed subsets E, F are *separated by a continuous function* if there is a map $f : X \to [0,1]$ such that $E \subseteq f^{-1}[\{0\}]$ and $F \subseteq f^{-1}[\{1\}]$.

Lemma

Let X be a T_1 space. Then X is normal if and only if every disjoint closed subset can be separated by continuous functions.

QUESTION: We have that the product of two Hausdorff spaces is Hausdorff. Is the product of two normal spaces normal?

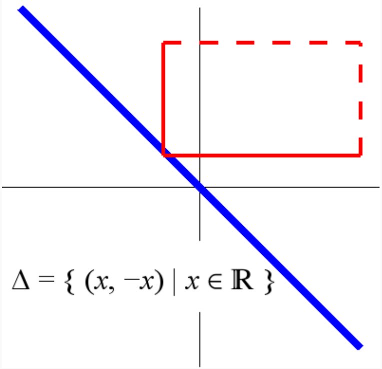

Figure 1: Sorgenfrey Plane

Summary

S. Class	Type of Sep.	Non-Ex.
T_0	$x \neq y$ then x, y are top. distinguishable	Ind. Top
T_1	$x \neq y$ then $\exists U \in \tau$, $x \in U \not\ni y$	Kripke
T_2	$x \neq y$ then $\exists U, V \in \tau$, $x \in U, y \in V, U \cap V = \emptyset$	Cof. Top.
T_4	T_2 + $E \cap F = \emptyset$, closed, $\exists U, V \in \tau$ $U \cap V = \emptyset$	Sorgenfrey

Next Lecture

- Saturated theories are definitive.
- Fluffy Filters and how to extend filters to fluffy filters.
- Compactness of topological spaces.
- Beginning of compactifications.

Thank you!
Questions?