Introduction to Topology in and via Logic
Lecture 4

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Plan for the Day

- Announcements
- Introduction to Filters and Filter Convergence.
- Hausdorff Spaces.
- Weaker Separation Axioms.
- Stronger Separation Axioms.

Announcements

1. Missing two people/teams.
2. Need to reorganize the teams for the two dates of presentation.

Recap

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Today we will focus on the property of separation.

Recap (from Soren's call and Amity's tutorial)

Definition

Let X be a set. We say that a collection of subsets $F \subseteq \mathcal{P}X - \{\varnothing\}$ is a *filter base* if it satisfies the following:

- $X \in F$;
- If $A, B \in F$ then $A \cap B \in F$.

We say that a given filter base is a *filter* if it is upwards closed: whenever $A \in F$ and $A \subseteq B$ then $B \in F$.

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Motivating example on Cantor space, see blackboard.

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Definition

Let (X, τ) be a topological space and $F \subseteq \tau$ a filter base. We say that the filter base F *converges to a point* x, and that x is the *limit of the filter base*, if and only if for every $U \in \mathcal{N}(x)$, there is some $V \in F$ such that $V \subseteq U$.

Separation

Separation: A Logical Example

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An Interesting Property: If I have two distinct maximally consistent theories Γ, Δ, there should be some ϕ such that $\phi \in \Gamma$ and $\phi \notin \Delta$. Topologically, there is some open A_ϕ such that $\Gamma \in A_\phi$ and $\Delta \notin A_\phi$. This means we can separate the two theories.

Definition

Let (X, τ) be a topological space. We say that X is *Hausdorff*, or T_2, if whenever $x, y \in X$ and $x \neq y$, there there exist two open neighbourhoods $x \in U_x$ and $y \in V_y$ and

$$U_x \cap V_y = \varnothing.$$

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Hausdorff Space

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Example

Any set with the discrete topology. All but one set with the *indiscrete* topology are *not* Hausdorff.
The Cantor space is Hausdorff and so is the Real Line.

Theorem

Let (X, τ) be a topological space. Then the following are equivalent:

1. *X is Hausdorff;*
2. *For each filter base F, F converges to at most one point;*

Proof: See Blackboard

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A note about constructions: products, coproducts and subspaces preserve Hausdorff; quotients in general do not.

Break: $\pm$ 10 minutes.

Definition

Let (X, τ) be a topological space. We say that X is *Frèchet* or T_1 if for all $x \neq y \in X$ there exists an open neighbourhood U_x such that $x \in U_x$ and $y \notin U_x$.

Example

Cofinite topology on a set; special case: Zariski topology on $\mathbb{R}$. See blackboard.

Definition

Let (X, τ) be a topological space. We say that two points x, y are *topologically distinguishable* if there exists an open neighbourhood $U_{x,y}$ such that either $x \in U_{x,y}$ and $y \notin U_{x,y}$ or $y \in U_{x,y}$ and $x \notin U_{x,y}$. We say that the space X is T_0 if all pairs of points are topologically distinguishable.

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Example

Consider $\mathbb{F} = (W, R)$ a Kripke frame. Then one can prove the following two facts, related to these weak separation properties:

1. The topological space induced by $\mathbb{F}$ is T_1 is and only if R corresponds to the identity.
2. The topological space induced by $\mathbb{F}$ is T_0 if and only if R is a partial order (i.e., it is antisymmetric).

In a sense, this means that *only T_0 spaces are interesting for epistemic settings where we want to represent different degrees of knowledge explicitly.*

Stronger Separation axioms: Normal

Definition

Let (X, τ) be a Hausdorff topological space. We say that X is *normal* or T_4 if whenever E, F are disjoint closed sets, then there exist open sets U, V, $E \subseteq U$ and $F \subseteq V$, such that $U \cap V = \varnothing$.

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Definition

Let X be a topological space. We say that two disjoint closed subsets E, F are *separated by a continuous function* if there is a map $f : X \to [0,1]$ such that $E \subseteq f^{-1}[\{0\}]$ and $F \subseteq f^{-1}[\{1\}]$.

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Lemma

Let X be a T_1 space. Then X is normal if and only if every disjoint closed subset can be separated by continuous functions.

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Consider the topology τ_{Sor} on $\mathbb{R}$ given by the following basis for $a < b \in \mathbb{R}$:

$$[a, b) = \{x : a \leqslant x < b\}$$

Then $\tau_{Euc} \subseteq \tau_{Sor}$. This is called the Sorgenfrey line.

Proposition

The Sorgenfrey line is normal.

Now consider the product of two copies of the Sorgenfrey line:

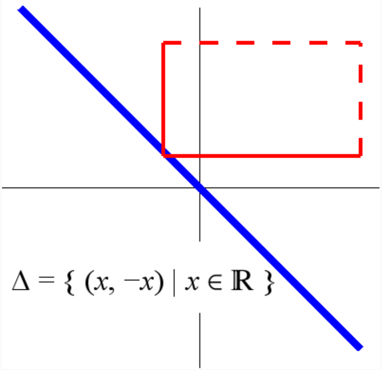

Figure 1: Sorgenfrey Plane

Stronger Separation Axioms: Clopen

Going back to the example of the logic space, this space is a peculiar property: it is not just separated, but separated by clopens.

Definition

Let (X, τ) be a topological space. We say that X is *totally separated* if whenever $x, y \in X$ are distinct points, there is a clopen set U such that $x \in U$ and $y \notin U$.

Example

The canonical topological space of the logic is totally separated. The Cantor space is totally separated.

Every totally separated space is T_1. As we will see in the coming lectures, when more topological properties are present, totally separated spaces can have very special properties.

S. Class	Type of Sep.	Non-Ex.
T_0	$x \neq y$ then x, y are top. distinguishable	Ind. Top
T_1	$x \neq y$ then $\exists U \in \tau$, $x \in U \not\ni y$	Kripke
T_2	$x \neq y$ then $\exists U, V \in \tau$, $x \in U, y \in V$, $U \cap V = \varnothing$	Cof. Top.
T_4	T_2 + $E \cap F = \varnothing$, closed, $\exists U, V \in \tau$ $U \cap V = \varnothing$	Sorgenfrey

Tomorrow

- Fluffy Filters and how to extend filters to fluffy filters.

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- Compactifications.

Thank you!
Questions?