Introduction to Topology in and via Logic
Lecture 5

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Plan for the Day

- Announcements
- Fluffy Filters
- Compactness in spaces.
- Compactness and Filters.
- Compact Hausdorff Spaces.
- Introduction to compactifications.

Recap

Definition

Let X be a set. We say that a collection of subsets $F \subseteq \mathcal{P}X - \{\varnothing\}$ is a *filter base* if it satisfies the following:

- $X \in F$;
- If $A, B \in F$ then $A \cap B \in F$.

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Definition

Let (X, τ) be a topological space and $F \subseteq \tau$ a filter base. We say that the filter base F *converges to a point* x, and that x is the *limit of the filter base*, if and only if for every $U \in \mathcal{N}(x)$, there is some $V \in F$ such that $V \subseteq U$.

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Today we look at the converse.

An Epistemic Example

To understand the idea we start with a concrete epistemic example. Suppose that $X = \omega$ is a set of worlds, indexed by the natural numbers. We think of the world n as a world where an alarm clock, somewhere deep in a forest in Madagascar, goes off exactly n seconds after 3:00 on January 17, 2023.

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This theory seems to hint at something: The alarm is broken.

But our model has no way to represent this possibility!

Fluffy Filters

To model this we need special kinds of filters:

Definition

Let X be a set and F a filter. We say that F is a *prime filter* (or sometimes a fluffy filter[a]) if it satisfies the following:

- For each $S \subseteq X$, either $S \in F$ or $X - S \in F$.

[a] See the end notes of Chapter 7 for an explanation of this nomenclature.

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Example: for each x, the collection $F(x)$

$$\{U \subseteq X : x \in U\}$$

is a prime filter. But in general there could be many more!

Theorem (Tarski, Prime Filter Theorem)

Let X be a set and F a filter base on X. Then there exists a prime filter $G \supseteq F$.

Proof: A proof is in the notes. We will not do it here, as the technique is mostly lattice theoretic, and we will only need the statement (but in MSL this is proven in full).

Compactness

We now have enough to discuss another crucial topological property.

Definition

Let X be a topological space, and $A \subseteq X$. Given a collection $(U_i)_{i \in I}$ of open sets, we say that this is an open cover of A if:

$$A = \bigcup_{i \in I} U_i.$$

Given such a cover, we say that a subcollection $(U_j)_{j \in J}$ for $J \subseteq I$ is a *subcover* if it is a cover of A. We say that a cover is *finite* if I is finite.

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Definition

Let X be a topological space. We say that X is compact if whenever $(U_i)_{i \in I}$ is an open cover, there exists a finite $I_0 \subseteq I$ such that $(U_j)_{j \in I_0}$ is a subcover of X.

Example

Negative example: $\mathbb{R}$ is not compact.

Positive example: the Cantor space is compact. See blackboard.

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Proposition

Let (X, τ) be a topological space with a basis $\mathcal{B}$. Then a set is compact if and only if every cover of X by basic open sets has a finite subcover.

Proof: See Blackboard.

What about subbases?

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What about subbases?

Theorem (Alexander Subbase Theorem)

Let X be a topological space with a subbasis $\mathcal{S}$. Then X is compact if and only if every cover of X by subbasic opens has a finite subcover.

Let us reformulate our criterion for compactness a bit:

Definition

Let X be a set, and $S \subseteq \mathcal{P}X$ a family of subsets. We say that S has the *finite intersection property* if whenever $A_0, ..., A_n \in S$ then $A_0 \cap ... \cap A_n \neq \varnothing$.

Lemma

Let X be a topological space. Then X is compact if and only if, whenever $\mathcal{F}$ is a family of closed subsets with the finite intersection property, then $\bigcap \mathcal{F} \neq \varnothing$.

Proof: See Blackboard.

Break: $\pm$ 10 minutes.

Proposition

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Proof: See Blackboard.

Hence we have that in compact spaces we can think of saturated theories as being the definitive ones.

Compact Subsets

A very important theme when dealing with compact spaces is the notion of a compact subspace.

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Lemma

Let X be a compact topological space. If A is closed, then A is a compact subset.

Proof: Exercise (worked out in the notes).

When we have compactness and Hausdorff separation, we can get in general much stronger results than with either property separate:

Theorem

Let X be a topological space. If X is compact and Hausdorff, then:

- *X is Normal;*
- *The compact subsets are precisely the closed ones.*
- *If $f : X \to Y$ is a continuous bijection from a compact to a Hausdorff space, then f is a homeomorphism.*

Proof: See Blackboard.

Compactification

Given the previous theorem, one would want to find some principled way of turning spaces into compact Hausdorff ones. In general, we already need some separation to start with. But we can add compactness.

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Definition

Let X, Y be topological spaces such that $f : X \to Y$ is a continuous function. We say that the pair (Y, f) is a *topological extension* of X if $f[X]$ is dense in Y (i.e., $\overline{f[X]} = Y$).[a]. We say that an extension is

- A *compactification*: if Y is compact;
- A *proper extension* if f is a homeomorphism and X is non-compact.
- A *strong compactification* if it is a proper extension, a compactification, and $f[X]$ is open in Y.

[a] Note: None of this terminology is standard, since the existing terminology seems to differ a lot between authors.

Example

Consider the space $\mathbb{N}$ with the discrete topology. Mo infinite discrete space is ever compact. But we can compactify this in a natural way:

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$$0 \qquad 1 \qquad 2 \qquad \ldots \qquad \omega$$

Figure 1: One-point compactification

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Figure 1: One-point compactification

We add a point and declare that a subset S is open in the new space if and only if it was already open, or it is a cofinite subset containing ω. Then this becomes a compact space, denoted $\alpha(\omega)$ or the Alexandroff compactification of the naturals.

Next Lecture

- Connectedness.

Next Lecture

- Connectedness.
- Disconnectedness

Next Lecture

- Connectedness.
- Disconnectedness
- Stone spaces.

Next Lecture

- Connectedness.
- Disconnectedness
- Stone spaces.
- Scattered spaces.

Next Lecture

- Connectedness.
- Disconnectedness
- Stone spaces.
- Scattered spaces.
- The End?

Thank you!
Questions?