Introduction to Topology in and via Logic

Lecture 5

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Plan for the Day

- Fluffy Filters
- Compactness in spaces.
- Compactness and Filters.
- Compact Hausdorff Spaces.
- Introduction to compactifications.

Recap

Theorem

Let (X, τ) be a topological space. Then the following are equivalent:

1. *X is Hausdorff;*
2. *For each filter base F, F converges to at most one point;*

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But what about existence?

A Logical Intuition

Recall our canonical topological space $X = MCS$, whose elements are maximally consistent sets, with the basis given by:

$$\{A_\phi : \phi \text{ a formula } \} \text{ where } A_\phi = \{x \in MCS(P) : \phi \in x\}.$$

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Now suppose that S is a consistent set of formulas which is closed under conjunction. Let Γ_S be given as

$$\Gamma_S = \{A_\phi : \phi \in S\}.$$

Then Γ_S is a filter base. By Lindenbaum's lemma, we can extend S to a maximally consistent set of formulas, i.e., there is some $x \in X$ such that $x \in \bigcap \Gamma_S$.

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Key Property: if we have a filter base, then it can be extended to a filter base which converges to a point. The way to go about this goes via fluffy filters.

To model this we need special kinds of filters:

Definition

Let X be a set and F a filter. We say that F is a *prime filter* (or sometimes a fluffy filter[a]) if it satisfies the following:

- For each $S \subseteq X$, either $S \in F$ or $X - S \in F$.

[a] See the end notes of Chapter 7 for an explanation of this nomenclature.

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Example: for each x, the collection $F(x)$

$$\{U \subseteq X : x \in U\}$$

is a prime filter. But in general there could be many more!

Extending Filters

Theorem (Tarski, Prime Filter Theorem)

Let X be a set and F a filter base on X. Then there exists a prime filter $G \supseteq F$.

Proof: A proof is in the notes. We will not do it here, as the technique is mostly lattice theoretic, and we will only need the statement (but in MSL this is proven in full).

We now have enough to discuss another crucial topological property.

Definition

Let X be a topological space, and $A \subseteq X$. Given a collection $(U_i)_{i\in I}$ of open sets, we say that this is an open cover of A if:

$$A = \bigcup_{i\in I} U_i.$$

Given such a cover, we say that a subcollection $(U_j)_{j\in J}$ for $J \subseteq I$ is a *subcover* if it is a cover of A. We say that a cover is *finite* if I is finite.

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Definition

Let X be a topological space. We say that X is compact if whenever $(U_i)_{i \in I}$ is an open cover, there exists a finite $I_0 \subseteq I$ such that $(U_j)_{j \in I_0}$ is a subcover of X.

Proposition

Let (X, τ) *be a topological space with a basis* $\mathcal{B}$*. Then a set is compact if and only if every cover of* X *by basic open sets has a finite subcover.*

Proof: See Blackboard.

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Positive example: the Cantor space is compact. See blackboard.

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What about subbases?

Theorem (Alexander Subbase Theorem)

Let X be a topological space with a subbasis $\mathcal{S}$. Then X is compact if and only if every cover of X by subbasic opens has a finite subcover.

Let us reformulate our criterion for compactness a bit:

Definition

Let X be a set, and $S \subseteq \mathcal{P}X$ a family of subsets. We say that S has the *finite intersection property* if whenever $A_0, ..., A_n \in S$ then $A_0 \cap ... \cap A_n \neq \varnothing$.

Lemma

Let X be a topological space. Then X is compact if and only if, whenever $\mathcal{F}$ is a family of closed subsets with the finite intersection property, then $\bigcap \mathcal{F} \neq \varnothing$.

Proof: See Blackboard.

Proposition

Let (X, τ) *be a topological space. Then* X *is compact if and only if whenever* $\mathcal{U}$ *is a fluffy filter, then* $\mathcal{U}$ *converges in* X.

Proof: See Blackboard.

Break: $\pm$ 10 minutes.

Compact Subsets

A very important theme when dealing with compact spaces is the notion of a compact subspace.

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Lemma

Let X be a compact topological space. If A is closed, then A is a compact subset.

Proof: Exercise (worked out in the notes).

When we have compactness and Hausdorff separation, we can get in general much stronger results than with either property separate:

Theorem

Let X be a topological space. If X is compact and Hausdorff, then:

- *X is Normal;*
- *The compact subsets are precisely the closed ones.*
- *If $f : X \to Y$ is a continuous bijection from a compact to a Hausdorff space, then f is a homeomorphism.*

Proof: See Blackboard.

Compactification

Given the previous theorem, one would want to find some principled way of turning spaces into compact Hausdorff ones. In general, we already need some separation to start with. But we can add compactness.

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Definition

Let X be a topological space. We say that a set $A \subseteq X$ is dense if $\overline{A} = X$.

Definition

Let X, Y be topological spaces such that $f : X \to Y$ is a continuous function. We say that the pair (Y, f) is a *topological extension* of X if $f[X]$ is dense in Y. We say that an extension is a decent compactification if:

- Y is compact;
- f is a homeomorphism;
- X is non-compact.
- $f[X]$ is open in Y.

(Guess: Not all compactifications will be decent, but we will not be too worried!)

Example

Consider the space $\mathbb{N}$ with the discrete topology. No infinite discrete space is ever compact. But we can compactify this in a natural way:

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Figure 1: One-point compactification

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$$0 \qquad 1 \qquad 2 \qquad \ldots \qquad \omega$$

Figure 1: One-point compactification

We add a point and declare that a subset S is open in the new space if and only if it was already open, or it is a cofinite subset containing ω. Then this becomes a compact space, denoted $\alpha(\omega)$ or the Alexandroff compactification of the naturals.

Next Lecture

- Connectedness.

Next Lecture

- Connectedness.
- Disconnectedness

Next Lecture

- Connectedness.
- Disconnectedness
- Stone spaces.

Next Lecture

- Connectedness.
- Disconnectedness
- Stone spaces.
- Scattered spaces.

Next Lecture

- Connectedness.
- Disconnectedness
- Stone spaces.
- Scattered spaces.
- The End?

Thank you!
Questions?