Introduction to Topology in and via Logic
Lecture 6

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January 25, 2023

Plan for the Day

- Announcements
- Compactifications.
- Connectedness.
- Disconnectedness.
- Extremal Disconnectedness.
- The End?

Definition

Let X be a set and F a filter. We say that F is a *prime filter* (or sometimes a fluffy filter[a]) if it satisfies the following:

- For each $S \subseteq X$, either $S \in F$ or $X - S \in F$.

[a] See the end notes of Chapter 7 for an explanation of this nomenclature.

Theorem (Tarski, Prime Filter Theorem)

Let X be a set and F a filter base on X. Then there exists a prime filter $G \supseteq F$.

Recap

Definition

Let X be a topological space. We say that X is compact if whenever $(U_i)_{i \in I}$ is an open cover, there exists a finite $I_0 \subseteq I$ such that $(U_j)_{j \in I_0}$ is a subcover of X.

Theorem

Let X be a topological space. If X is compact and Hausdorff, then:

- *X is Normal;*
- *The compact subsets are precisely the closed ones.*
- *If $f : X \to Y$ is a continuous bijection from a compact to a Hausdorff space, then f is a homeomorphism.*

Recap: Compactifications

Definition

Let X, Y be topological spaces such that $f : X \to Y$ is a continuous function. We say that the pair (Y, f) is a *topological extension* of X if $f[X]$ is dense in Y (i.e., $\overline{f[X]} = Y$).[a]. We say that an extension is

- A *compactification*: if Y is compact;
- A *proper extension* if f is a homeomorphism and X is non-compact.
- A *strong compactification* if it is a proper extension, a compactification, and $f[X]$ is open in Y.

[a] Note: None of this terminology is standard, since the existing terminology seems to differ a lot between authors.

We also gave the example of $\alpha(\omega)$. We will now take a look at a more general instance of the latter kind of example.

Definition

Let X be a topological space. Let $X^* := X \sqcup \{\infty\}$, and topologise this as follows: a subset $U \subseteq X^*$ is open either if it is open in X, or if $U = X - C \cup \{\infty\}$ where C is a compact and closed subset of X.

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Proposition

Let X be a non-compact topological space. Then (X^*, i) is a strong compactification of X.

Proof: See Blackboard.

The former is most useful when, in a technical sense, the space is already compact on a "small scale":

Definition

Let X be a Hausdorff space. We say that X is *locally compact* if for each $x \in X$ there is a compact neighbourhood of x.

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Proof: Exercise 5.8 in the notes. A counterexample is also added to other plausible sounding conjectures.

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Can we find a different, perhaps more canonical solution?

Definition

Let X be a topological space. We say that a pair (Y, i) where $i : X \to Y$ is a *Stone-Cech compactification* if it satisfies the following property: if Z is a compact and Hausdorff space, and $f : X \to Z$ is a continuous function, there is a unique continuous function $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ i$.

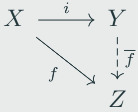

Figure 1: Stone-Cech Compactification

Stone-Cech compactifications

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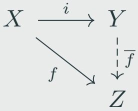

Figure 1: Stone-Cech Compactification

Observe: the construction is unique if it exists.

In general, this construction is not very easy to obtain or visualise. But it leads to important examples:

Example

Example of $\beta\omega$, the Stone-Cech compactification of the naturals, and $\beta\omega - \omega$, the Parovicenko space.

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One can also construct this for general spaces, but we leave that task for the brave reader wishing to venture in that part of the notes.

Example

Compactifications of duals of products of algebras: example with Boolean algebras.

Connectedness

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The key difference is that instead of requiring the proposition to be verifiable, we straight up ask it to be decidable. This seems like a plausible requirement.

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If I have two points x, y*, there should be a line uniting the two of them.*

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Can we satisfy the logician and the geometer in interesting spaces? No.

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Let (X, τ) be a topological space. We say that X is *connected* if the only clopen subsets of X are X and $\varnothing$.

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This can be reformulated:

Proposition

Let (X, τ) be a topological space. Then X is connected if and only if the only continuous functions $f : X \to \{0, 1\}$ are constant.

Proof: Exercise.

Example

The real line $\mathbb{R}$ is connected. The Cantor space is not connected (we will see this later).

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But does it work?

Definition

Let X be a topological space. We say that X is *path-connected* if whenever $x, y \in X$, there is some path p from x to y.

Proposition

Let X be a path-connected space. Then X is connected.

Proof: See Blackboard.

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Example

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Example

Example of $\mathbb{R} - \{0\}$ and $\mathbb{R}^2 - \{0\}$.

Shocked at the intuitive mismatch between the former, we rush to make our spaces as little connected as possible. We come with the following:

Definition

Let X be a topological space. We say that a subset $A \subseteq X$ is a *connected component* if A is connected, and whenever $A \subsetneq B \subseteq X$, then B is not connected. We denote by $Con(X)$ the set of connected components of X.

Definition

Let X be a topological space. We say that X is *totally disconnected* if whenever $A \subseteq X$ and A is connected, then there is $x \in X$ such that $A = \{x\}$.

Just like in the case for connectedness, one can come up with a different definition which arguably fits the epistemologist better:

Definition

Let X be a topological space. Given two points $x, y \in X$, we write $x \equiv_{QC} y$ if and only if for all clopens $U \subseteq X$, $x \in U$ if and only if $y \in U$.
We say that X is *totally separated* if $x \equiv_{QC} y$ if and only if $x = y$.

Lemma

Let X be a topological space. Then:

1. *If X is totally separated, then X is totally disconnected.*
2. *If X is compact and Hausdorff, the converse also holds.*

Proof: See Blackboard.

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Proof: See Blackboard.

Example

The Cantor space is totally separated. See Blackboard.

An extremely important example of totally separated spaces are Stone Spaces.

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Let X be a topological space. Then X is a Stone space if and only if it is a compact Hausdorff space generated by a basis of clopen subsets.

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Example

The Stone-Cech compactification of any discrete space is a Stone space.

Disconnectedness: A Lost Promise

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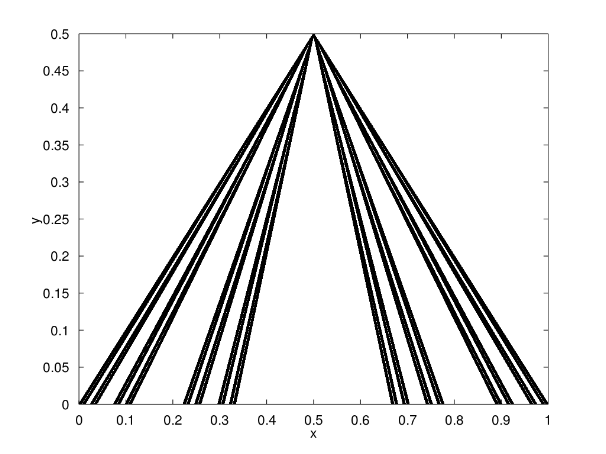

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- Isolated points.
- Extremally disconnected spaces.
- Scattered spaces.
- I encourage you to have a read of these concepts, and try to do some exercises, as they are also quite ubiquitous in logic; but unfortunately we do not have time for it all!

Thank you!
Questions?