Topic Presentation Ideas

Below you will find a number of topic presentation ideas. These are organised by section, and contain a brief outline of what the general theme is, and some ideas of how to structure your presentation. More precise ideas can be discussed later.

General Topological Questions

Whilst this course has covered many topics you will see in an introductory course in topology, even in the domain of general topology some natural concepts were left out. Hence it might be interesting to deepen these concepts.

- **Keeping you Close: Metric Spaces**: One of the key aspects of the real line is that you can keep track of the distance between two points on a space. Indeed, for products of the real line, the same holds. This is basically down to *metric spaces*. Introduce the notion of a metric space and show that all metric spaces are topological spaces. Discuss basic properties of metric spaces: topology, sequences in metric spaces, sequential continuity. Introduce the notion of a complete metric space, and define the completion of a metric space. In case you have time (probably if $\geq$3 pax), you can look into the question of when a topological space is a metric space: give an example of a topological space that is not metrizable, and use Urysohn's Lemma to prove his metrization theorem or go even further by characterising the class of metrizable topological spaces via the Nagata-Smirnov Metrization Theorem or the Smirnov Metrization Theorem.

- **Keeping you Close: Uniformities**: Sometimes the notion of a metric can be too strong for some applications. A weaker notion might be needed which nevertheless still captures the idea that two points are "close" in a strong, but qualitative, rather than quantitative sense. Define uniform spaces. Relate uniform spaces to metric and topological spaces. Then choose one: introduce uniform completions and Cauchy filters; or show that a topology is Tychonoff if and only if it can be given a uniform structure.

- **Keeping you Close: Order Topologies**: We have seen that when defining the standard (Euclidean) topology on $\mathbb{R}$, all we need to consider is its order: we generate the topology via the basis consisting of all open intervals. This idea generalises most naturally to constructing a topology on any given linear order, called its *order topology*, which allows for generalising various theorems from analysis. Introduce the notion of an order topology and show how these topologies interact with other basic notions we have seen. For instance, you can show (some of) the following: (i) for convex subsets, restricting the order topology is the same as taking the subspace topology; (ii) all order topologies are Hausdorff (and even normal); (iii) all convex subsets of dense linear orders with the l.u.b. property are connected; (iv) all connected sets in the order topology are dense and have the l.u.b. property; (v) if X, Y are order topologies and $f : X \to Y$ is order-preserving and surjective, then f is a homeomorphism; and (vi) all order topologies are regular. Then prove (1) a generalised topological intermediate value theorem and (2) a generalised topological extreme value theorem, and show how the usual Intermediate Value Theorem and Extreme Value Theorem on $\mathbb{R}$ obtain as special cases.

- **Projecting on Compact Hausdorff**: We have seen that the compact Hausdorff spaces are very nice structures. Investigate (two out of the three) further properties of these objects: given two compact Hausdorff spaces X, Y, there is a topology that can be put on X^Y, called the compact-open topology; a space is called *compactly generated* if it is coherent with the family of its compact subspaces – show they are all topological sums of compact Hausdorff spaces; show that if P is a compact Hausdorff and extremally disconnected space, and $f : P \to X$ is a continuous map, and $e : E \to X$ is a surjective continuous map, then there is a map $\overline{f} : P \to E$ such that $\overline{f} \circ e = f$.

Set Theoretic Topology

What we have covered in this project is very much the core theory of general topology, but there was a set of topics which, for matters of space, were left out – the cardinality invariants, and other analysis of space which depend on the existence of specific combinatorial structures.

- **Weighing the Character of a Space**: On par with the separation axioms, we have axioms controlling the size of a space, and cardinal invariants corresponding to them. Introduce the notion of "weight", "density" and "character" of a topological space. Define first-countable, separable and second-countable spaces; give examples and counterexamples for these notions. Define the class of regular spaces. Then prove Urysohnn's embedding theorem.

- **Suslin's Real Problem**: Axiomatising the reals has been a challenge throughout the history of mathematics, and many attempts were made to make precise what is special about them. The intimate connection of the reals with linear orders allows one to ask questions about this. Define a linearly ordered topological space. Provide a proof of the classical result showing that a linear order is complete, separable, dense and unbounded if and only if it is the real line. Introduce Suslin's problem. Introduce Martin's Axiom, and show that it resolves the Suslin problem in the positive.

- **Bor(r)eling in the Real Line**: When working with topological spaces, not all sets are open or closed. However, some sets can still be obtained by, in some sense "accessible operations". Describe the concept of a Borel set, and the corresponding Borel algebra. Introduce the notion of a Polish space as a natural setting to discuss Borel sets. Then prove that the Borel hierarchy does not collapse. If interested, discuss the connections with computability theory.

- **The Mysteries of** $\beta\omega$: The space $\beta\omega$, and the remainder $\beta\omega - \omega$ constitute a very complex structure, described as a "beast with three heads". Discuss p-points and their independence, to illustrate this. If interested, you can relate these filters to Ramsey-like theorems, where they appear naturally. Then provide (Engelking's topological) proofs of Parovicenko's theorems (without all details): in the presence of CH the space $\beta\omega - \omega$ is the unique space with the following properties:

  1. Compact Hausdorff, which is crowded;
  2. Has weight equal to the cardinality of the continuum;
  3. Every two disjoint countable unions of closed sets have disjoint closures;
  4. Every non-empty countable intersection of open sets has non-empty interior.

  And in fact, this characterization is *equivalent* to CH.

Algebraic Topology

One aspect which is somewhat unsatisfactory about the notion of continuity is that, whilst it captures the idea that transformations are in some sense unbroken, this intuition is not truly captured by continuity. The most intuitive idea of continuity – of a continuous "deformation" of space, of a folding of space – can only be captured by other, more sophisticated concepts. These happen to almost always involve some algebra.

- **It is simple**[**x**]: Simplicial complexes constitute one of the simplest, yet most versatile classes of spaces available in mathematics, and they are used in everything, from modal logic, to category theory, to algebraic topology and set theory. Give the definition of a simplicial complex, and the basic properties of these spaces. Then define the basic idea of a chain group, boundaries and cycles, and show how this implements the idea of "higher-dimensional holes". Give many examples.

- **Deforming Space**: Homotopies are the right notion of continuous deformation of space. They can be given the structure of a *group*, which leads to the powerful methods of algebra. Introduce the notion of a homotopy, and give examples. Define the fundamental group. Define a covering map, and prove that $\mathbb{Z}$ is the fundamental group of the circle. If there is time, give a proof of the Borsuk-Ulam Theorem for $n = 2$.

Logical Topology

We started this course by discussing an epistemic interpretation of topology, and later found that a very natural modal logic implements precisely the idea of being an interior operator, an equivalent way of defining topologies. How far does the logical description of topology go?

- **Shower of Semantics**: It seems clear that **S4** is the modal logic of all topological spaces, and also the modal logic of the real line. However, there are more modal logics associated to space. Discuss (some of) the following: the d-semantics of modal logic; intuitionistic logic as a logic of space; Provability logic and scattered spaces; amongst others.

- **Hyperspaces and Beyond**: A classic result by Leo Esakia constructed a form of "topological Kripke frames" by considering topologies on the power set of a topological space – so called "hyperspaces". Give an outline of the Vietoris topology on a general space, and its consequence for Stone spaces: the Vietoris constructions maps Stone spaces to Stone spaces. Show that there is a 1 to 1 correspondence between specific maps to the Vietoris space and descriptive general frames.

Categorical Topology

Our whole emphasis in this course has been highly categorical. We have made use of an analogy – topology as a space of learning, or computability – which is strongly preffered by category theorists and computer scientists alike. However, we have strived to keep the presentation close to a classical presentation, mostly through a strong concern with point-set notions. What happens if you relinquish that?

- **Let's get Stoned**: Often we think of topological spaces as sets with extra structure; however, it is also possible, in certain cases, to think of them as special kinds of algebras. Introduce Stone duality through the lens of capturing space through algebra, and contrast with Stone's original motivation (capturing algebra through topology). Then discuss the basics of locale theory: completely prime filters, spatial locales and sober spaces.

- **(Topping Topoi)**: Topoi constitute a wide range of structures with wide applications. But even their definition can sometimes elicit fear in the heart of people. Define a sheaf over topological spaces, and illustrate, with examples of sheaves of continuous functions, that these are natural structures. Then define a (localic) Grothendieck topos as this kind of structure. Give a brief sketch of the sheafification procedure.

Philosophy and Topology

In our approach we emphasised the intuitions of epistemology to motivate our work. Many of these intuitions can be made more thoroughly connected through some formalisation of this in the form of topological epistemic logic. But this does not exhaust the connections between philosophy and topology. Mereology has emerged in the 21st century as a source of metaphysical questions, and as a deeply sophisticated formal theory, often rich in mathematical subtlety alongside its philosophical interest. In this setting, theories of location have given rise to specific approaches that share with topology an interest in space, occuppying space, amongst other spatial questions.

- **(Misleading Defeaters and Defeating Misleaders)**: Some recent work has connected epistemic logic and classic discussions about knowledge, justified belief, defeasibility and misleadingness. Present the basic arguments of the paper by Baltag, Bezhanishvili, Ozgun and Smets, and present the proposals of some topological models. Then analyse the logic. You should pick one particular epistemic concept – evidence, justified belief, knowledge – to focus on.

- **(Space is a Soup)**: Region-based theories of spatial structure are formalisations of space that depend on the notion of a "region" rather than points. These are often "thick" structures, and quite distinct from point-set topological spaces. Discuss the formalisation by Casati and Varzi, providing examples and applications to philosophy of your interest.

- **(Gunk up the Punk)**: Gunk is a possible kind of object which contains no atomic parts. Indecomposable objects are structures which admit no mereological decomposition into disjoint parts. Though mostly the subject of philosophical debate, in the origins of contemporary topology, the axiomatisation of the real line, involved debates concerning whether this line ought to be gunky or indecomposable. In this presentation, you should explore the consequences of the existence of these kinds of objects, with a focus on an intuitionistic perspective of the real line. Present the classical arguments for the gunkiness and indecomposability of the real line, and some proofs of these on the basis of decidability. Then discuss how certain classical objects allow us to have a classically compatible idea of these facts: locales as "generalised spaces", region-based theories of space, and indecomposable continua.