

## TOPOLOGY IN AND VIA LOGIC HOMEWORK ASSIGNMENT 1

- Deadline: February 09 at 23:59.
- All exercises are worth the same points.
- Good luck!

**Exercise 1.** Consider the space  $(\mathbb{R}, \tau_{Euc})$ , with its Euclidean topology.

- (1) Give an example of a set which is neither open nor closed.
- (2) Show that the open intervals of the form  $(x, y)$  where  $x, y \in \mathbb{Q}$  form a basis for this topology.
- (3) Show that  $\mathbb{Q}$  is a countable union of closed sets.

**Exercise 2.** Let  $X$  be a set. We say that an operation  $\square : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  is called an *interior operator* if it satisfies for each  $U, V \in \mathcal{P}(X)$ ,

- (All set):  $\square X = X$ ;
- (Normality):  $\square(U \cap V) = \square U \cap \square V$ ;
- (Inflationarity):  $\square U \subseteq U$ ;
- (Idempotence):  $\square U \subseteq \square \square U$ .

- (1) Show that if  $(X, \tau)$  is a topological space, the topological interior  $int$  is an interior operator in this sense.
- (2) Given a set  $(X, \square)$  equipped with an interior operator, define a topology for which  $\square$  is the topological interior operator.
- (3) We say that an interior operator  $\square$  is *completely multiplicative* if for each  $(U_i)_{i \in I}$  we have that:

$$\square\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} \square U_i$$

Show that Alexandroff topologies are in 1-1 correspondence with completely multiplicative interior operators.

- (4) Let  $(X, \square)$  be a set, equipped with a completely multiplicative interior operator, with the following property: if  $x \neq y$ , then there is some  $U \subseteq X$  such that either  $x \in \square U$  and  $y \notin \square U$  or  $y \in \square U$  and  $x \notin \square U$ . Show that then there is a poset  $(P, \leq)$  such that the Alexandroff topology on  $P$  is the same as the topology induced on  $X$  by the interior operator.

**Exercise 3.** Let  $(X, \tau)$  be a topological space. We say that a map  $\nu : \tau \rightarrow \tau$  is a *nucleus* if it satisfies the following for all open subsets  $U \subseteq X$ :

- (i)  $U \subseteq \nu(U)$ ;

- (ii) If  $U \subseteq V$  then  $\nu(U) \subseteq \tau(V)$ ;
- (iii)  $\nu(\nu(U)) = \nu(U)$ ;
- (iv)  $\nu(U \cap V) = \nu(U) \cap \nu(V)$ .

(1) Show that if  $K \subseteq X$  is any subset, then the map  $\nu_K : \tau \rightarrow \tau$  given by setting

$$\nu_K(U) = \text{int}([X - K] \cup U)$$

for all opens  $U$ , is a nucleus, called the *induced nucleus of  $K$* .

(2) (\*) Note that the map:

$$j_{\neg\neg}(U) = \text{int}(cl(U))$$

is a nucleus as well. Show that there is a topological space  $(X, \tau)$  such that  $j_{\neg\neg}$  on this space is not the induced nucleus of any set  $K \subseteq X$ . *Hint: Consider the real line and show that the only  $K$  which could induce such a nucleus is the empty set, and that the empty set does not induce it.*