TOPOLOGY IN AND VIA LOGIC
HOMEWORK ASSIGNMENT 2

- Deadline: February 10 at 23:59.
- All exercises are worth the same points.
- Good luck!

Separation

Exercise 1. Let (X, τ) be a topological space. Let $A \subseteq X$ be a subset. We say that A is *dense in* X if $\overline{A} = X$.

(1) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.
(2) Show that if A is dense in X, then A intersects all open subsets of X.
(3) Let X be a topological spaces and Y a Hausdorff space, and let $A \subseteq X$ be a dense subset. Let $f, g : X \to Y$ be two continuous functions such that $f{\restriction}_A = g{\restriction}_A$. Show that $f = g$.
(4) Show that if A is dense in X, then for each $x \in X$, there is a filter over A (i.e., $F \subseteq \mathcal{P}(A)$) converging to x.

Exercise 2. Show the following for a T_1-space X:

(1) If X is finite, then the topology on it is discrete.
(2) For each $x \in X$, $\{x\}$ is closed.
(3) For each $x \in X$, the filter
$$F(x) := \{S \subseteq X : x \in S\}$$
converges uniquely to x.

Show that the last property is an alternative definition for T_1-spaces.

Compactness

Exercise 3. Let X be a compact Hausdorff space, and A a closed subspace. Define the equivalence relation $x \sim y$ if and only if either $x = y$ or x and y are both in A. Show that the quotient space $X/{\sim}$ is compact Hausdorff.

Exercise 4. Let X and X_i throughout be Hausdorff spaces.

(1) Show that every compact space is locally compact.
(2) Show that the converse inclusion does not hold *Hint: Think of* $\mathbb{R}$.
(3) Show that if X is any Hausdorff and non-compact space, then $\alpha(X)$ is compact Hausdorff if and only if X is locally compact.

Connectedness

Exercise 5. Prove Proposition 6.1.4 of the lecture note. *Hint: The cardinality of the interval* $[0, 1]$ *is exactly* $2^{\aleph_0}$.