TOPOLOGY IN AND VIA LOGIC
HOMEWORK ASSIGNMENT 2

- Deadline: February 09 at 23:59.
- All exercises are worth the same points.
- Good luck!

Continuity and Continuous Functions

Exercise 1. Show the following:

(1) Give an example of a bijective continuous map which is not a homeomorphism.

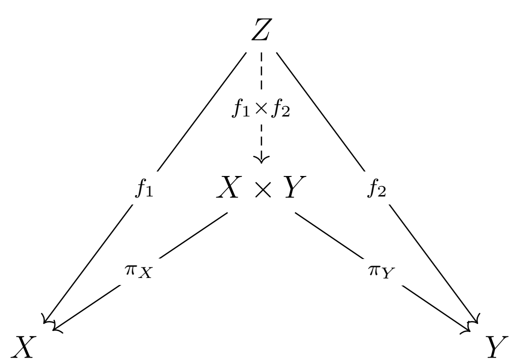
(2) Show that all functions from a discrete space to another space are continuous. If (X, τ) is a space with the *indiscrete* topology, which functions from this space to some other space are continuous?

(3) Show that if $f : X \to Y$ is a bijective continuous map between topological spaces, then the following are equivalent:
- f^{-1} is continuous;
- f is closed;
- f is a homeomorphism.

Exercise 2. (*Product maps*) Let X, Y be topological spaces.

(1) Show that the product topology is the coarstest topology on the set $X \times Y$ such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.

(2) Show that for any other topological space Z, if there exist continuous functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, then there exists a unique continuous function $f_1 \times f_2 : Z \to X \times Y$ making the following diagram commute



(3) Show that this defines the product topology up to homeomorphism: whenever a topological space A together with two continuous functions $\pi_{A,X} : A \to X$ and $\pi_{A,Y} : A \to Y$ satisfy the condition in (1), then there exists a homeomorphism between A and $X \times Y$. *Hint: Given topological spaces X, Y continuous map $f : X \to Y$ is*

a homeomorphism if and only if there is a continuous map $g : Y \to X$ such that $fg = id_Y$ and $gf = id_X$.

Separation

Exercise 3. Let (X, τ) be a topological space. Let $A \subseteq X$ be a subset. We say that A is *dense in* X if $\overline{A} = X$.

(1) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.
(2) Show that if A is dense in X, then A intersects all open subsets of X.
(3) Let X be a topological spaces and Y a Hausdorff space, and let $A \subseteq X$ be a dense subset. Let $f, g : X \to Y$ be two continuous functions such that $f\restriction_A = g\restriction_A$. Show that $f = g$.
(4) Show that if A is dense in X, then for each $x \in X$, there is a filter over A (i.e., $F \subseteq \mathcal{P}(A)$) converging to x.

Exercise 4. Show the following for a T_1-space X:

(1) If X is finite, then the topology on it is discrete.
(2) For each $x \in X$, $\{x\}$ is closed.
(3) For each $x \in X$, the filter

$$F(x) := \{S \subseteq X : x \in S\}$$

converges uniquely to x.

Show that the last property is an alternative definition for T_1-spaces.