Domain Theory

Alyssa Renata, Paulius Skaisgiris, Sarah Dukic, Joel Saarinen

February 1, 2023

Table of Contents

Table of Contents

Denotational Semantics

- Origins of Domain Theory lie in 1970s, work by Dana Scott and Christopher Strachey

Denotational Semantics

- Origins of Domain Theory lie in 1970s, work by Dana Scott and Christopher Strachey
- Denotational semantics: assigning "meaning" to a given program/expression or a datatype in a programming language.

Denotational Semantics

- Origins of Domain Theory lie in 1970s, work by Dana Scott and Christopher Strachey
- Denotational semantics: assigning "meaning" to a given program/expression or a datatype in a programming language.
- More formally: given a programming language P, for each datatype D (eg. expressions, commands, integers, etc.) in that language, there is a valuation function v that maps a phrase of syntax in that category to a denotation in a semantic structure D - the **domain** of interpretation.

Denotational semantics

Example

- Consider the following program:
  def $g(x)\{\text{return } x + 4 \overset{?}{=} 8\}$
- $g(x) \in \{0, 1\}$, so regardless of input, we can come up with a denotation for g.

Denotational Semantics for datatypes themselves

Syntax of (some) datatypes

$$
\begin{array}{rl}
\mathit{Command} ::= & \mathbf{if}\ \mathit{Bool}\ \mathbf{then}\ \mathit{Command}\ \mathbf{else}\ \mathit{Command} \\
\mid & \mathbf{while}\ \mathit{Bool}\ \mathbf{do}\ \mathit{Command} \mid \mathbf{def}\ \mathrm{x}\ \mathbf{:=}\ \mathit{Value} \mid \mathbf{run}\ \mathrm{x} \\
\mid & \mathit{Command}\ \mathbf{;}\ \mathit{Command} \mid \mathbf{skip}
\end{array}
$$

$\mathit{Bool} ::= \mathbf{tt} \mid \mathbf{ff} \mid \mathrm{x} \mid \mathit{Bool}\ \mathbf{and}\ \mathit{Bool} \mid \mathit{Bool}\ \mathbf{or}\ \mathit{Bool} \mid \ldots$

$\mathit{Int} ::= \mathbf{0} \mid \mathbf{1} \mid \ldots \mid \mathrm{x} \mid \textbf{-}\ \mathit{Int} \mid \mathit{Int}\ \mathbf{+}\ \mathit{Int} \mid \ldots$

$\mathit{Value} ::= \mathit{Bool} \mid \mathit{Int} \mid \mathit{Command}$

Semantics of (some) datatypes

$[\![\mathit{Command}]\!] = \mathit{State} \to \mathit{State}$

$\mathit{State} = \mathit{Vars} \to [\![\mathit{Value}]\!]$

$[\![\mathit{Bool}]\!] = \mathbb{B}$

$[\![\mathit{Int}]\!] = \mathbb{Z}$

$[\![\mathit{Value}]\!] = [\![\mathit{Bool}]\!] \bigcup [\![\mathit{Int}]\!] \bigcup [\![\mathit{Command}]\!]$

Issues with denotational semantics: infinitely looping functions

- What happens if we try to associate a function/value with the following program?

Issues with denotational semantics: infinitely looping functions

- What happens if we try to associate a function/value with the following program?
- $f : \mathbb{N} \to \mathbb{N},\ f(x) = f(x) + 1$
-
```
def f(x) {
    return f(x) + 1
}
```

Issues with denotational semantics: infinitely looping functions

- What happens if we try to associate a function/value with the following program?
- $f : \mathbb{N} \to \mathbb{N},\ f(x) = f(x) + 1$
- ```
  def f(x) {
        return f(x) + 1
  }
  ```
- It will loop and not map to a single number:

$$
\begin{aligned}
f(m) &= f(m) + 1 \\
f(m) &= f(m) + 1 + 1 \\
f(m) &= f(m) + 1 + 1 + 1
\end{aligned}
$$
  ```

Issues with denotational semantics: recursively generated semantic spaces

- A similar problem arises when we try to come up with semantics for recursively generated structures. For example, as the valuation for the datatype **value** is defined as follows:
  $[\![Value]\!] = [\![Bool]\!] \cup [\![Int]\!] \cup [\![Command]\!]$
  where
  $[\![Command]\!] = State \rightarrow State$
  $State = Vars \rightarrow [\![Value]\!]$
  $[\![Bool]\!] = \mathbb{B}$
  $[\![Int]\!] = \mathbb{Z}$

Issues with denotational semantics: recursively generated semantic spaces

- A similar problem arises when we try to come up with semantics for recursively generated structures. For example, as the valuation for the datatype **value** is defined as follows:
  $[\![Value]\!] = [\![Bool]\!] \cup [\![Int]\!] \cup [\![Command]\!]$
  where
  $[\![Command]\!] = State \rightarrow State$
  $State = Vars \rightarrow [\![Value]\!]$
  $[\![Bool]\!] = \mathbb{B}$
  $[\![Int]\!] = \mathbb{Z}$
- Since one of the terms in finding the meaning of the datatype Value requires us to find the meaning of Value again, the same procedure is repeated indefinitely, not settling upon a single interpretation.
- Moreover, if: $|state| = n$ then $|state \rightarrow state| = n^n$.

Criteria for a solution: fixpoints

- Observation: every recursively defined function can be expressed as a non-recursive function.

Criteria for a solution: fixpoints

- Observation: every recursively defined function can be expressed as a non-recursive function.
- Take $f : \mathbb{N} \to \mathbb{N}, f(x) = f(x) + 1$ as seen in the previous slides. We can define a non-recursive higher-order function Φ, where $\Phi(f) = z \mapsto f(z) + 1$. We can then rewrite f as $f = \Phi(f)$.
- If we let $\Phi : A \to A$, then $x \in A$ is called a *fixed point* of Φ if $\Phi(x) = x$.

Criteria for a solution: fixpoints

- Observation: every recursively defined function can be expressed as a non-recursive function.
- Take $f : \mathbb{N} \to \mathbb{N}, f(x) = f(x) + 1$ as seen in the previous slides. We can define a non-recursive higher-order function Φ, where $\Phi(f) = z \mapsto f(z) + 1$. We can then rewrite f as $f = \Phi(f)$.
- If we let $\Phi : A \to A$, then $x \in A$ is called a *fixed point* of Φ if $\Phi(x) = x$.
- Idea: element in semantic space that a given recursive function is mapped to could be the fixed point of the non-recursive function Φ that we rewrite f in terms of.
- However, a function might have no fixpoints, or rather several - so how do we define a denotational semantics that captures these cases?

Fixpoints for recursively generated semantic spaces

- Recall: finding $[\![Value]\!] := [\![Bool]\!] \bigcup [\![Int]\!] \bigcup [\![Command]\!]$ involves finding $[\![Command]\!]$. But $[\![Command]\!] = state \to state$, where

$$state = var \to [\![Value]\!] \tag{1}$$

- We can rewrite (1) as: $state = var \to [\![Bool]\!] \bigcup [\![Int]\!] \bigcup state \to state$

Fixpoints for recursively generated semantic spaces

- Recall: finding $[\![Value]\!] := [\![Bool]\!] \bigcup [\![Int]\!] \bigcup [\![Command]\!]$ involves finding $[\![Command]\!]$. But $[\![Command]\!] = state \rightarrow state$, where

$$state = var \rightarrow [\![Value]\!] \tag{1}$$

- We can rewrite (1) as: $state = var \rightarrow [\![Bool]\!] \bigcup [\![Int]\!] \bigcup state \rightarrow state$
- Problem simplifies to:

$$state \cong state \rightarrow state \tag{2}$$

Fixpoints for recursively generated semantic spaces

- We showed that we arrive at the above difficulty by trying to evaluate the meaning of Value. However, one encounters similar difficulties with semantics of other datatypes, having to deal with equations similar to (2), only with mathematical constructions other than "state".
- Ultimately, we seek to solve:

$$D \cong D \to D$$

Evolution of the datatype

$$\begin{tikzcd} \text{datatype} \arrow[r] & ? \arrow[r] & ? \arrow[d] \\ \begin{array}{c}\text{Non-trivial}\\ \text{solution to}\\ D \cong [D \to D]\end{array} & & ? \arrow[ll] \end{tikzcd}$$

Table of Contents

Proposed solution: partial functions

- Problem: we cannot use total functions to map between datatypes because there are functions with no fixpoints.
- Solution: use partial functions. We can take progressively better finite approximations of our infinitely recurring function, and take the limit of these.

Example

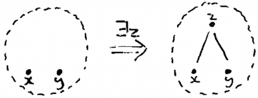
See blackboard.

The structure of datatypes: DCPO's

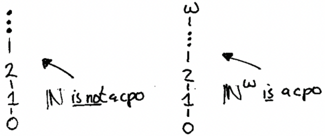
1. A datatype is partially ordered.
   - Want to represent that one datatype might contain the same information as another. $f \sqsubseteq g$ captures the intuition that g is a consistent extension of f.
   - Set theoretically, $f \sqsubseteq g \iff f \subseteq g$. So g can compute what f can, and more: e.g. $f = \{(0,1),(1,2)\}$ and $g = \{(0,1),(1,2),(2,3)\}$

The structure of datatypes: DCPO's

2. Datatypes are directed complete, with a bottom element.
   - A subset $X \subseteq D$ where any two points $x, y \in X$ have an upper bound $z \in X$.



   - We want our datatypes to have consistent specifications of information.
   - For any directed subset, we want an element containing all its information: a least upper bound.
   - Thus, every directed subset of a datatype has a least upper bound. Our datatypes are therefore **directed complete partial orders**.

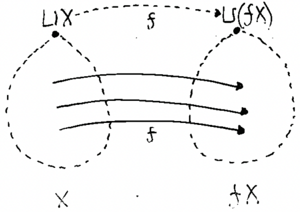


Mappings on datatypes

3. Mappings between datatypes are monotonic.
   - A function $f : D' \to D$ should be sensitive to the accuracy of the input.
   - Consider $\phi(f)$ versus $\phi(g)$ where $f := \{(0,1),(1,1)\}$ and $g := \{(0,1),(1,1),(2,2)\}$. Then $\phi(g)$ is defined whenever $\phi(f)$ is defined, but the converse is not true. So $f \sqsubseteq g \to \phi(f) \sqsubseteq \phi(g)$

Mappings on datatypes

4. Mappings between datatypes are continuous.
   - We want functions to preserve limits: "finite" information in the output should entail "finite" information in the input.
   - LUB's of directed sets should be preserved:
     $f : D \to D'$ is continuous iff $f(\bigsqcup X) = \bigsqcup\{f(x) : x \in X\}$



   - Continuity gives us the **DCPO fixpoint theorem:** Any continuous function $f : A \to A$ on a DCPO A has a LFP computed as the limit of $\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \dots$, i.e. $LFP(f) = \bigsqcup\{f^n(\bot) | n \in \mathfrak{N}\}$

Summary

- We now have the desired structure for datatypes. DCPO's both capture our intuitions about how information should behave, and also gives us a way of specifying the denotation of recursive functions non-recursively.
- The presence of a bottom element lets us characterise functions with no output
- Computable functions are monotonic and continuous.
- The DCPO fixpoint theorem tells us that there is always an LFP: we have a denotation for any function.

Evolution of the datatype

$$
\begin{tikzcd}
\text{datatype} \arrow[r] & \text{DCPO} \arrow[r] & ? \arrow[d] \\
\begin{array}{c}\text{Non-trivial}\\ \text{solution to}\\ D \cong [D \to D]\end{array} & & ? \arrow[ll]
\end{tikzcd}
$$

Table of Contents

Topological Intuitions

- Any computable function must be *monotone* and *preserves directed joins*.

Topological Intuitions

- Any computable function must be *monotone* and *preserves directed joins*.
- Informally, directed joins $=$ limits, so join-preservation $=$ continuity.

Topological Intuitions

- Any computable function must be *monotone* and *preserves directed joins*.
- Informally, directed joins = limits, so join-preservation = continuity.
- Can we make this analogy formal?

Alexandroff Topology

Reminder

Given partial order $(P, \leq)$, U is open in the Alexandroff topology on P iff U is upwards closed (if $x \in U$ and $x \leq y$ then $y \in U$).

Alexandroff Topology

Reminder

Given partial order $(P, \leq)$, U is open in the Alexandroff topology on P iff U is upwards closed (if $x \in U$ and $x \leq y$ then $y \in U$).

- Alexandroff-continuity = monotone

Alexandroff Topology

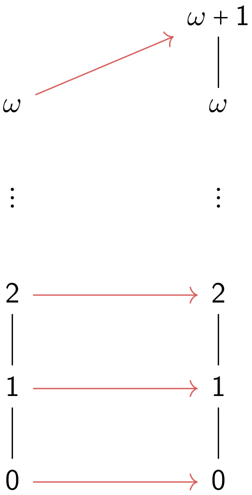
Reminder

Given partial order $(P, \leq)$, U is open in the Alexandroff topology on P iff U is upwards closed (if $x \in U$ and $x \leq y$ then $y \in U$).

- Alexandroff-continuity = monotone
- But what about join-preservation?

Not all Alexandroff-continuous functions preserve d-joins

Consider the following monotone function from $\omega + 1$ to $\omega + 2$:

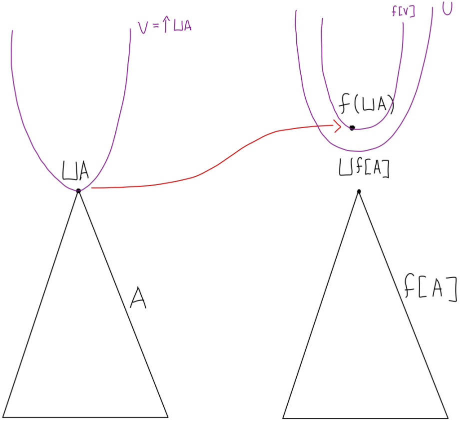


Not all Alexandroff-continuous functions preserve d-joins

General Problem: Even if $f(\bigsqcup A) \neq \bigsqcup f[A]$, f can still be continuous around $f(\bigsqcup A)$ - for any open neighborhood U of $f(\bigsqcup A)$, the image of the open neighborhood $V = \uparrow \bigsqcup A$ lies in U.

Not all Alexandroff-continuous functions preserve d-joins

General Problem: Even if $f(\bigsqcup A) \neq \bigsqcup f[A]$, f can still be continuous around $f(\bigsqcup A)$ - for any open neighborhood U of $f(\bigsqcup A)$, the image of the open neighborhood $V = \uparrow \bigsqcup A$ lies in U.



The Remedy

The Remedy

Want: For every open neighborhood V of $\bigsqcup A$, $f[V]$ must contain $\bigsqcup f[A]$.

The Remedy

Want: For every open neighborhood V of $\bigsqcup A$, $f[V]$ must contain $\bigsqcup f[A]$. **Solution**: Force every such V to contain an element of A.

The Remedy

Want: For every open neighborhood V of $\bigsqcup A$, $f[V]$ must contain $\bigsqcup f[A]$. **Solution**: Force every such V to contain an element of A.

Definition (Scott Topology)

Let D be a DCPO. We define the **Scott topology** σ_D *on D by defining the* **Scott-open** *sets as follows.*

1. *$U \in \sigma_D$ iff (i) U is upwards closed and (ii) for any directed set A, if $\bigsqcup A \in U$ then $U \cap A$ is non-empty.*
2. *When U satisfies condition (ii) above, we say that U is* **inaccessible by directed joins**.

The Remedy

Want: For every open neighborhood V of $\bigsqcup A$, $f[V]$ must contain $\bigsqcup f[A]$. **Solution**: Force every such V to contain an element of A.

Definition (Scott Topology)

Let D be a DCPO. We define the **Scott topology** σ_D *on D by defining the* **Scott-open** *sets as follows.*

1. *$U \in \sigma_D$ iff (i) U is upwards closed and (ii) for any directed set A, if $\bigsqcup A \in U$ then $U \cap A$ is non-empty.*
2. *When U satisfies condition (ii) above, we say that U is* **inaccessible by directed joins**.

Proposition

S is **Scott-closed** *iff S is downwards closed and closed under directed joins.*

The Scott Topology

- Intuition: open set $=$ finitely observable property. Hence, if we can finitely observe a property of $\bigsqcup A$ then this property should already be evident in some component $a \in A$ of $\bigsqcup A$.

The Scott Topology

- Intuition: open set = finitely observable property. Hence, if we can finitely observe a property of $\bigsqcup A$ then this property should already be evident in some component $a \in A$ of $\bigsqcup A$.
- The following theorems formalise the idea that directed joins = limits and preservation of directed joins = continuity.

Theorem

1. *Let A be a directed set in DCPO D. Then in the Scott topology σ_D, $\bigsqcup A$ is a limit of the filter generated by closing $\{\uparrow a \cap A \mid a \in A\}$ upwards.*

Proof.

See blackboard. □

The Scott Topology

- Intuition: open set = finitely observable property. Hence, if we can finitely observe a property of $\bigsqcup A$ then this property should already be evident in some component $a \in A$ of $\bigsqcup A$.
- The following theorems formalise the idea that directed joins = limits and preservation of directed joins = continuity.

Theorem

1. *Let A be a directed set in DCPO D. Then in the Scott topology σ_D, $\bigsqcup A$ is a limit of the filter generated by closing $\{\uparrow a \cap A \mid a \in A\}$ upwards.*
2. *$f : D \to E$ is continuous under the Scott topology iff it is monotone and preserves directed joins.*

Proof.

See blackboard. □

Example of Scott Topology: $\mathbb{N} \rightharpoonup \mathbb{N}$

- The Scott topology on the DCPO of partial functions is generated by subbasic opens of the form $\uparrow \{(m, n)\}$.

Example of Scott Topology: $\mathbb{N} \rightharpoonup \mathbb{N}$

- The Scott topology on the DCPO of partial functions is generated by subbasic opens of the form $\uparrow \{(m, n)\}$.
- Closure under intersections give upsets of all finite partial functions:

$$\uparrow \{(m_0, n_0), \ldots (m_k, n_k)\} = \uparrow \{(m_0), n_0)\} \cap \ldots \cap \uparrow \{(m_k, n_k)\}$$

Example of Scott Topology: $\mathbb{N} \rightharpoonup \mathbb{N}$

- The Scott topology on the DCPO of partial functions is generated by subbasic opens of the form $\uparrow \{(m, n)\}$.
- Closure under intersections give upsets of all finite partial functions:

$$\uparrow \{(m_0, n_0), \ldots (m_k, n_k)\} = \uparrow \{(m_0), n_0)\} \cap \ldots \cap \uparrow \{(m_k, n_k)\}$$

- Closure under union generates all the Scott open sets, but notice that we never end up generating upsets of infinite partial functions - any Scott open set contains a finite partial function.

Table of Contents

Up-sets are no longer open

- In general, $\uparrow x$ is not open in the Scott Topology.

Up-sets are no longer open

- In general, $\uparrow x$ is not open in the Scott Topology.
- As a substitute, we can still take the interior $int(\uparrow x)$.

Up-sets are no longer open

- In general, $\uparrow x$ is not open in the Scott Topology.
- As a substitute, we can still take the interior $int(\uparrow x)$.
- Intuition: $y \in int(\uparrow x)$ means x is relatively small (often this even means finite) compared to y - i.e. x is wayyy below y.

Up-sets are no longer open

- In general, $\uparrow x$ is not open in the Scott Topology.
- As a substitute, we can still take the interior $int(\uparrow x)$.
- Intuition: $y \in int(\uparrow x)$ means x is relatively small (often this even means finite) compared to y - i.e. x is wayyy below y.

Definition

x **is way below** y *(denoted* $x \ll y$*) iff for any directed* A, $y \sqsubseteq \bigsqcup A$ *implies there is some* $a \in A$ *s.t.* $x \sqsubseteq a$.

Up-sets are no longer open

- In general, $\uparrow x$ is not open in the Scott Topology.
- As a substitute, we can still take the interior $int(\uparrow x)$.
- Intuition: $y \in int(\uparrow x)$ means x is relatively small (often this even means finite) compared to y - i.e. x is wayyy below y.

Definition

x **is way below** y *(denoted* $x \ll y$*) iff for any directed* A, $y \sqsubseteq \bigsqcup A$ *implies there is some* $a \in A$ *s.t.* $x \sqsubseteq a$.

Proposition

$y \in int(\uparrow x)$ *implies* $x \ll y$. *In particular, if* $\bigsqcup A \in int(\uparrow x)$ *then there is some* $a \in A$ *such that* $x \sqsubseteq a$.

Up-sets are no longer open

- In general, $\uparrow x$ is not open in the Scott Topology.
- As a substitute, we can still take the interior $int(\uparrow x)$.
- Intuition: $y \in int(\uparrow x)$ means x is relatively small (often this even means finite) compared to y - i.e. x is wayyy below y.

Definition

x **is way below** y *(denoted $x \ll y$) iff for any directed A, $y \sqsubseteq \bigsqcup A$ implies there is some $a \in A$ s.t. $x \sqsubseteq a$.*

Proposition

$y \in int(\uparrow x)$ implies $x \ll y$. In particular, if $\bigsqcup A \in int(\uparrow x)$ then there is some $a \in A$ such that $x \sqsubseteq a$.

Example

Consider the DCPO of partial functions on $\mathbb{N}$. Then $\{(0,0),(1,2)\} \ll (x \mapsto 2x)$ but $(x \text{ even} \mapsto 2x) \not\ll (x \mapsto 2x)$.

Compact Elements

Compact Elements

When $x \ll x$, then x is a small approximation of itself, which is possible only if x is small in some absolute sense.

Compact Elements

When $x \ll x$, then x is a small approximation of itself, which is possible only if x is small in some absolute sense.

Definition

If $x \ll x$, we say that x is **compact***. If D is a DCPO, let $D_c = \{x \in D | x \ll x\}$ be the set of its compact elements.*

Compact Elements

When $x \ll x$, then x is a small approximation of itself, which is possible only if x is small in some absolute sense.

Definition

If $x \ll x$, we say that x is **compact***. If D is a DCPO, let $D_c = \{x \in D | x \ll x\}$ be the set of its compact elements.*

Example

- Any finitely defined partial function is compact.

Compact Elements

When $x \ll x$, then x is a small approximation of itself, which is possible only if x is small in some absolute sense.

Definition

If $x \ll x$, we say that x is **compact**. *If D is a DCPO, let $D_c = \{x \in D | x \ll x\}$ be the set of its compact elements.*

Example

- Any finitely defined partial function is compact.
- If D is a finite or flat DCPO, $D = D_c$.

Compact Elements

When $x \ll x$, then x is a small approximation of itself, which is possible only if x is small in some absolute sense.

Definition

If $x \ll x$, we say that x is **compact**. *If D is a DCPO, let $D_c = \{x \in D | x \ll x\}$ be the set of its compact elements.*

Example

- Any finitely defined partial function is compact.
- If D is a finite or flat DCPO, $D = D_c$.
- In $\omega + 1$, only ω is not compact.

Compact Elements

When $x \ll x$, then x is a small approximation of itself, which is possible only if x is small in some absolute sense.

Definition

If $x \ll x$, we say that x is **compact**. *If D is a DCPO, let $D_c = \{x \in D | x \ll x\}$ be the set of its compact elements.*

Example

- Any finitely defined partial function is compact.
- If D is a finite or flat DCPO, $D = D_c$.
- In $\omega + 1$, only ω is not compact.
- More generally, in any ordinal DCPO $\alpha + 1$, the compact elements are the successor ordinals and 0.

Algebraicity

- If we are to interpret a datatype as a DCPO, then every element must be computable as a limit of finitely computable elements.

Algebraicity

- If we are to interpret a datatype as a DCPO, then every element must be computable as a limit of finitely computable elements.
- Compactness is an abstraction of being finitely computable.

Algebraicity

- If we are to interpret a datatype as a DCPO, then every element must be computable as a limit of finitely computable elements.
- Compactness is an abstraction of being finitely computable.

Axiom (Algebraicity)

A datatype must have a "basis" of compact elements: for each $x \in D$*, the set* $approx(x) = \downarrow x \cap D_c$ *must be directed with* $x = \bigsqcup approx(x)$*.*

Algebraicity

- If we are to interpret a datatype as a DCPO, then every element must be computable as a limit of finitely computable elements.
- Compactness is an abstraction of being finitely computable.

Axiom (Algebraicity)

A datatype must have a "basis" of compact elements: for each $x \in D$*, the set* $approx(x) = \downarrow x \cap D_c$ *must be directed with* $x = \bigsqcup approx(x)$*.*

Example

Many of the standard DCPOs such as the DCPO of partial functions etc. are algebraic.

Algebraic DCPOs are determined by their compact elements

Proposition

1. *Let D and E be algebraic DCPOs. Then $f : D \to E$ is continuous iff $f(x) = \bigsqcup f[approx(x)]$.*
2. *Let D and E be DCPOs with D algebraic. Each monotone function $f : D_c \to E$ extends uniquely to a continuous $\overline{f} : D \to E$.*

Evolution of the datatype

$$
\begin{tikzcd}
\text{datatype} \arrow[r] & \text{DCPO} \arrow[r] & \text{algebraic DCPO} \arrow[d] \\
\text{Non-trivial solution to } D \cong [D \to D] & & ? \arrow[ll]
\end{tikzcd}
$$

Table of Contents

Semantics for function types is hard

- Clearly, having function types is useful as it provides semantics for *command*

Semantics for function types is hard

- Clearly, having function types is useful as it provides semantics for *command*
- This was a source of difficulty because the amount of all possible functions (n^n) is far more than the amount of states we have (n)

Semantics for function types is hard

Now that we have some tools and definitions, we can attempt to fully untangle this problem. What we want:

- "Carve out" computable functions as these functions will be executed on a physical machine
  - We know that computable functions are monotone and continuous

Semantics for function types is hard

Now that we have some tools and definitions, we can attempt to fully untangle this problem. What we want:

- "Carve out" computable functions as these functions will be executed on a physical machine
  - We know that computable functions are monotone and continuous
- We need the set of all functions (function space) between two datatypes to also be a datatype

Semantics for function types is hard

Now that we have some tools and definitions, we can attempt to fully untangle this problem. What we want:

- "Carve out" computable functions as these functions will be executed on a physical machine
  - We know that computable functions are monotone and continuous
- We need the set of all functions (function space) between two datatypes to also be a datatype
  - Thus far, datatype = algebraic DCPO

Semantics for function types is hard

Now that we have some tools and definitions, we can attempt to fully untangle this problem. What we want:

- "Carve out" computable functions as these functions will be executed on a physical machine
  - We know that computable functions are monotone and continuous
- We need the set of all functions (function space) between two datatypes to also be a datatype
  - Thus far, datatype = algebraic DCPO
  - So, remains to show that for two algebraic DCPOs D, E, the set of all monotone and continuous functions from D to E, $[D \to E]$, is an algebraic DCPO.

Semantics for function types is hard

Now that we have some tools and definitions, we can attempt to fully untangle this problem. What we want:

- "Carve out" computable functions as these functions will be executed on a physical machine
  - We know that computable functions are monotone and continuous
- We need the set of all functions (function space) between two datatypes to also be a datatype
  - Thus far, datatype = algebraic DCPO
  - So, remains to show that for two algebraic DCPOs D, E, the set of all monotone and continuous functions from D to E, $[D \to E]$, is an algebraic DCPO.

Theorem (?)

Let D, E be algebraic DCPOs. Then, $[D \to E]$ is an algebraic DCPO.

What DCPO structure should $[D \to E]$ have?

- The set of all functions $D \to E$ is essentially the product $\Pi_{x \in D} E$ of D copies of E, so we can equip it with the product topology constructed from the Scott topology of each E.

What DCPO structure should $[D \to E]$ have?

- The set of all functions $D \to E$ is essentially the product $\Pi_{x \in D} E$ of D copies of E, so we can equip it with the product topology constructed from the Scott topology of each E.
- We consider the monotone and continuous subspace of $\Pi_{x \in D} E$, denoted $\int_{x \in D} E$.

What DCPO structure should $[D \to E]$ have?

- The set of all functions $D \to E$ is essentially the product $\Pi_{x \in D} E$ of D copies of E, so we can equip it with the product topology constructed from the Scott topology of each E.
- We consider the monotone and continuous subspace of $\Pi_{x \in D} E$, denoted $\int_{x \in D} E$.

Definition

The topological space $\int_{x \in D} E$ of **pointwise convergence** *on $[D \to E]$ is the topology generated by the basis*

$$\left\{ \int_{x \in D} U_x \mid \forall x \in D. U_x \in \sigma_E \right\}$$

with the condition that only finitely many $U_d \neq E$ in each $(U_x)_{x \in D}$.

What DCPO structure should $[D \to E]$ have?

- The set of all functions $D \to E$ is essentially the product $\Pi_{x \in D} E$ of D copies of E, so we can equip it with the product topology constructed from the Scott topology of each E.
- We consider the monotone and continuous subspace of $\Pi_{x \in D} E$, denoted $\int_{x \in D} E$.

Definition

The topological space $\int_{x \in D} E$ of **pointwise convergence** *on $[D \to E]$ is the topology generated by the basis*

$$\left\{ \int_{x \in D} U_x \mid \forall x \in D. U_x \in \sigma_E \right\}$$

with the condition that only finitely many $U_d \neq E$ in each $(U_x)_{x \in D}$.

Proposition

Given a filter F in $\int_{x \in D} E$, $F \to f$ iff $F(x) \to f(x)$ for each $x \in D$.

What DCPO structure should $[D \to E]$ have?

The order induced by the pointwise convergence topology on $[D \to E]$ yields the following DCPO:

Proposition

If D and E are DCPOs, then the partial order on $[D \to E]$ defined as

$$f \sqsubseteq g \iff \forall x \in D. f(x) \sqsubseteq g(x)$$

is a DCPO, with $(\bigsqcup F)(x) = \bigsqcup F(x)$.

What DCPO structure should $[D \to E]$ have?

The order induced by the pointwise convergence topology on $[D \to E]$ yields the following DCPO:

Proposition

If D and E are DCPOs, then the partial order on $[D \to E]$ defined as

$$f \sqsubseteq g \iff \forall x \in D. f(x) \sqsubseteq g(x)$$

is a DCPO, with $(\bigsqcup F)(x) = \bigsqcup F(x)$.

Unfortunately, it's **NOT** algebraic.

$[D \to E]$ is not necessarily algebraic

- A monotone continuous function $f : D \to E$ is constructed as the limit of compact approximations of the form

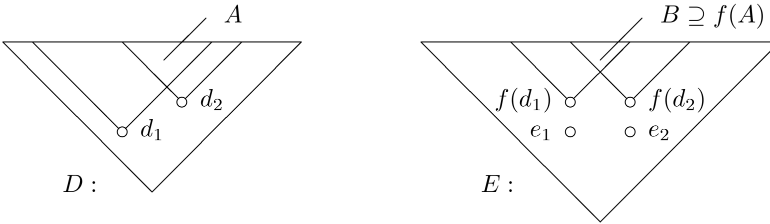
$$\langle d; e\rangle(x) := \begin{cases} e & \text{if } x \sqsubseteq d \\ \perp & \text{otherwise} \end{cases}$$

$[D \to E]$ is not necessarily algebraic

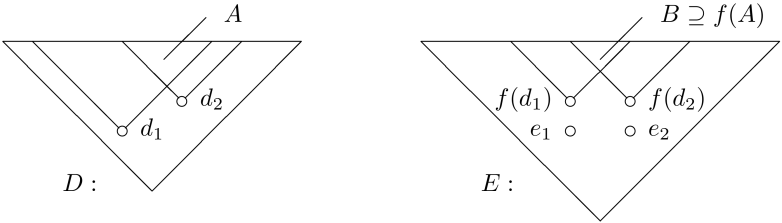
- A monotone continuous function $f : D \to E$ is constructed as the limit of compact approximations of the form

$$\langle d; e\rangle(x) := \begin{cases} e & \text{if } x \sqsubseteq d \\ \perp & \text{otherwise} \end{cases}$$

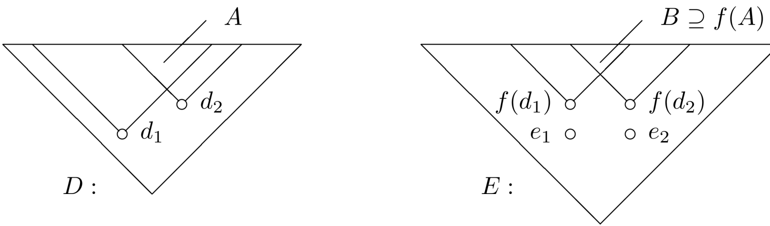
- However, the set of compact approximations *approx*(f) is not necessarily directed: consider how one constructs a compact upper bound of $\langle d_1; e_1\rangle$ and $\langle d_2; e_2\rangle$.



$[D \to E]$ is not necessarily algebraic

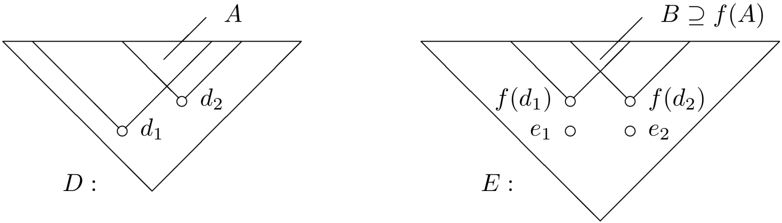


$[D \to E]$ is not necessarily algebraic



- The issue is that we cannot find what the upper bound of $\langle d_1; e_1 \rangle$ and $\langle d_2; e_2 \rangle$ should map to when given an element of $A = \uparrow d_1 \cap \uparrow d_2$, although ideally it would be $e_1 \sqcup e_2$.

$[D \to E]$ is not necessarily algebraic



- The issue is that we cannot find what the upper bound of $\langle d_1; e_1 \rangle$ and $\langle d_2; e_2 \rangle$ should map to when given an element of $A = \uparrow d_1 \cap \uparrow d_2$, although ideally it would be $e_1 \sqcup e_2$.
- e_1 and e_2 are arbitrary elements, other than the fact that they are upper bounded by $f(a)$ where a is some element of A. Hence, to fix this, we require the existence of certain additional least upper bounds.

Scott Domains

Definition

A partial order is **consistently complete** *iff every set with an upper bound has a least upper bound (join).*

Scott Domains

Definition

A partial order is **consistently complete** *iff every set with an upper bound has a least upper bound (join).*

Axiom (Consistently Complete)

A datatype has to be consistently complete.

Scott Domains

Definition

A partial order is **consistently complete** *iff every set with an upper bound has a least upper bound (join).*

Axiom (Consistently Complete)

A datatype has to be consistently complete.

This concludes our search for a suitable class of structures to represent domains of interpretation, for we are now in the position to solve the $D \cong [D \to D]$ equation.

Scott Domains

Definition

A partial order is **consistently complete** *iff every set with an upper bound has a least upper bound (join).*

Axiom (Consistently Complete)

A datatype has to be consistently complete.

This concludes our search for a suitable class of structures to represent domains of interpretation, for we are now in the position to solve the $D \cong [D \to D]$ equation.

Definition (Scott Domains)

A partial order is a (Scott) domain iff it is an **a***lgebraic,* **c***onsistently* **c***omplete, directed* **c***omplete* **p***artial* **o***rder with a bottom element.*

Scott Domains

Definition

A partial order is **consistently complete** *iff every set with an upper bound has a least upper bound (join).*

Axiom (Consistently Complete)

A datatype has to be consistently complete.

This concludes our search for a suitable class of structures to represent domains of interpretation, for we are now in the position to solve the $D \cong [D \to D]$ equation.

Definition (Scott Domains)

A partial order is a (Scott) domain iff it is an **a***lgebraic,* **c***onsistently* **c***omplete, directed* **c***omplete* **p***artial* **o***rder with a bottom element.*

a c3po →

Function space as a datatype revisited

Theorem
Let D, E be domains. Then the function space $[D \to E]$ is a domain.

Proof. By putting blind faith in us.
It works out, trust us bro.

Evolution of the datatype

$$
\begin{tikzcd}
\text{datatype} \arrow[r] & \text{DCPO} \arrow[r] & \text{algebraic DCPO} \arrow[d] \\
\begin{array}{c}\text{Non-trivial} \\ \text{solution to} \\ D \cong [D \to D]\end{array} & & \begin{array}{c}\text{consistently-complete} \\ \text{algebraic DCPO} \\ \text{(Scott Domains)}\end{array} \arrow[ll]
\end{tikzcd}
$$

Table of Contents

Domain equations

The last thing remaining is to find a general datatype to serve as a model for denotational computation. Essentially, we want our datatype to be a solution to the equation:

$$D \cong [D \to D]$$

Domain equations

The last thing remaining is to find a general datatype to serve as a model for denotational computation. Essentially, we want our datatype to be a solution to the equation:

$$D \cong [D \to D]$$

Note that the trivial solution is $D = \{\bot\}$, but, clearly, we want a more interesting solution.

Non-trivial solution: D_∞

Let D be a domain. Set $D_0 = D$ and define inductively D_n for each n by

$$D_{n+1} \cong [D_n \to D_n]$$

Non-trivial solution: D_∞

Let D be a domain. Set $D_0 = D$ and define inductively D_n for each n by

$$D_{n+1} \cong [D_n \to D_n]$$

"It turns out that there is a **natural** way of isomorphically embedding each D_n successively into the next space D_{n+1}" [Scott, 1970]

Non-trivial solution: D_∞

Let D be a domain. Set $D_0 = D$ and define inductively D_n for each n by

$$D_{n+1} \cong [D_n \to D_n]$$

"It turns out that there is a **natural** way of isomorphically embedding each D_n successively into the next space D_{n+1}" [Scott, 1970]

These embeddings allows us to take the limit of this equation, obtaining the limit space

$$D_\infty \cong [D_\infty \to D_\infty]$$

Non-trivial solution: D_∞

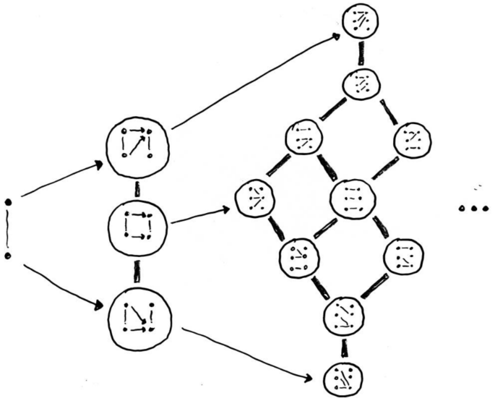
The construction idea starts by taking the limit of a sequence of domains obtained by iterating the function space construction.

That is, take $f = (f_0, f_1, f_2, \ldots)$. We may apply this function sequence onto itself getting $(f_1(f_0), f_2(f_1), f_3(f_2), \ldots)$.

Non-trivial solution: D_∞

This is a solution to the self-application problem because

- We have restricted the amount of functions considered to allow for $|D_\infty| = |[D_\infty \to D_\infty]|$



Non-trivial solution: D_∞

This is a solution to the self-application problem because

- We have restricted the amount of functions considered to allow for $|D_\infty| = |[D_\infty \to D_\infty]|$
- Each element of D_∞ can be regarded as a continuous function on D_∞ into D_∞, and every such continuous function can be regarded as an element.

Significance of D_∞

Quote

"Finding a non-trivial model of the untyped λ-calculus was Scott's original motivation for developing domain theory. The construction of such a model in 1972 is one of the most significant results in the history of theoretical computer science." [Hutton, 1994]

Quote

"Technically speaking, what we have here is the first known, 'mathematically' defined model of the so-called λ-calculus of Curry-Church." [Scott, 1970]

Table of Contents

Closing summary

- Domain theory is a field providing necessary tools for giving denotational semantics to programming languages (or, λ-calculus, generally)

Closing summary

- Domain theory is a field providing necessary tools for giving denotational semantics to programming languages (or, λ-calculus, generally)
- Partial orders with extra structure are chosen as the basic elements to represent datatypes, coined Scott Domains

Closing summary

- Domain theory is a field providing necessary tools for giving denotational semantics to programming languages (or, λ-calculus, generally)
- Partial orders with extra structure are chosen as the basic elements to represent datatypes, coined Scott Domains
- This order induces a topology which can be used as a supplement to order-theoretic treatment of the theory

Closing summary

- Domain theory is a field providing necessary tools for giving denotational semantics to programming languages (or, λ-calculus, generally)
- Partial orders with extra structure are chosen as the basic elements to represent datatypes, coined Scott Domains
- This order induces a topology which can be used as a supplement to order-theoretic treatment of the theory
- We illustrated a way to provide semantics to `command` by defining D_∞: a domain which is isomorphic to its function space

References

Hutton, G. (1994).
Introduction to domain theory.

Scott, D. (1970).
Outline of a mathematical theory of computation.
Oxford University Computing Laboratory, Programming Research Group Oxford.