Justified Belief and the Topology of Evidence

Presentation for "Topology in and via Logic"

Elias, Jonathan, Wouter

Universiteit van Amsterdam/ILLC

Introduction

Paper: *Justified Belief and the Topology of Evidence* (2016) by Alexandru Baltag, Nick Bezhanishvili, Aybüke Özgün and Sonja Smets.

1. Introduction
2. Formal definition of *evidence models*
3. Formal definition of *topo-e-models*
4. Arguments and justifications in topo-e-models
5. Operators for evidence, belief and knowledge
6. Philosophical assessment of the *K*-operator
7. Lehrer's defeasibility theory of knowledge & problems
8. Misleading evidence
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Evidence Models

Given a countable set of propositional letters Prop, an *evidence model* is a tuple $\mathcal{M} = (X, E_0, V)$, where:

- X is a non-empty set of states;
- $E_0 \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ is a family of non-empty sets called *basic evidence sets/pieces of evidence* s.t. $X \in E_0$;
- $V : Prop \to \mathcal{P}(X)$ is a valuation function.

Family $F \subseteq E_0$ of pieces of evidence is *consistent* if $\bigcap F \neq \emptyset$, inconsistent otherwise.

Body of evidence is a family $F \subseteq E_0$ s.t. every non-empty finite subfamily is consistent, i.e. F has the finite intersection property.

Body of evidence F *supports a proposition* P iff P is true in all worlds satisfying the evidence in F, i.e. $\bigcap F \subseteq P$.

Evidence Models cont'd

Strength order between bodies of evidence:

$F \subseteq F'$ means that F' is at least as strong as F. Stronger bodies of evidence support more propositions. A body of evidence is *maximal* if it's not included in any other body of evidence.

Combined evidence:

Any non-empty intersection of finitely many pieces of basic evidence, where E denotes the family of all combined evidence.

Support:

$e \in E$ *supports* a proposition/e is evidence for $P \subseteq X$ if $e \subseteq P$.

Strength order between combined evidence given by reverse inclusion: $e \supseteq e'$ means that e' is at least as strong as e.

Evidence and Factivity

$e \in E_0$ represent basic pieces of direct evidence (observation, testimony, etc.) possessed by the agent.
$e \in E$ represents indirect evidence obtained by combining pieces of direct evidence (evidence is not necessarily true).

$e \in E$ is *factive evidence* at world $x \in X$ iff e is true at x, i.e. $x \in e$. Similarly, a body of evidence F is factive if all the pieces of evidence $e \in F$ are factive, i.e. $x \in \bigcap F$.

Topological Evidence Models (topo-e-model)

Topology generated by $E \subseteq \mathcal{P}(X)$ is the smallest topology τ_E on X s.t. $E \subseteq \tau_E$.

$A \subseteq X$ is called *dense* in (X, τ) if $Cl(A) = X$ and it is called *nowhere dense* if $IntCl(A) = \emptyset$

A *topological evidence model* is a tuple $\mathcal{M} = (X, E_0, \tau, V)$, where (X, E_0, V) is an evidence model and $\tau = \tau_E$ (*evidential topology*) is the topology generated by the family of combined evidence E (basis) or by the family of basic evidence sets E_0 (subbasis).

Arguments, Justifications, and Factivity

An *argument for P* is a disjunction $U = \bigcup_{i \in I} e_i$ of evidences $e_i \in E$ that all support P, i.e. $e_i \subseteq P$ for all $i \in I$.
Topologically, an argument for P is a *non-empty open subset of P*, i.e. $U \in \tau_E$ s.t. $U \subseteq P$. $Int(P)$ is *the weakest (most general) argument for P*

A *justification for P* is an argument U for P which is consistent with every evidence, i.e. $U \cap e \neq \emptyset$ for all $e \in E$. Thus, justifications are arguments which are not defeated by any available evidence.
Topologically, a justification for P is an *(everywhere) dense open subset* of P, i.e. $U \in \tau_E$ s.t. $U \subseteq P$ and $Cl_{\tau_E}(U) = X$.

Argument or justification is *factive* if it is true in the actual world.

Justifications are the basis of *belief*, whereas correct justifications are the basis of *defeasible knowledge*

Loretta and her taxes

Example: Taxes

Loretta has done her taxes, careful to double check every calculation. Based on this evidence she correctly believes that she owes 500 Dollars.

O_1 : Loretta's direct evidence that she owes 500 Dollars. O_2 : Loretta's evidence that her accountant does not make mistakes in his replies.

$X = \{x_1, x_2, x_3, x_4, x_5\}$ and $E_0 = \{X, O_1, O_2\}$ where $O_1 = \{x_1, x_2, x_3\}$ and $O_2 = \{x_3, x_4, x_5\}$. Then $E = \{X, O_1, O_2, \{x_3\}\}$. Let x_1 be the actual world.

Generating a topology from E_0 or E gives us: $\tau_E = \{\emptyset, X, O_1, O_2, \{x_3\}\}$

Note: $Cl(O_1) = X$ and $x_1 \in Int(O_1) = O_1$, so O_1 is dense and it's an open neighbourhood of x_1. O_1 argument for itself, a justification, and it is factive

$Cl(O_2) = X$ but $x_1 \notin Int(O_2)$, so O_2 is dense as well, but it's not an open neighbourhood of x_1. O_2 argument for itself, a justification, but not factive.

Operators in Evidence Models

What can we do with all the previously introduced notions?

Introducing operators that (should) correspond to intuitive notions of knowledge/belief.

Infallible Knowledge

$\forall$ is a global modality. It associates to any proposition $P \subseteq X$ another proposition $\forall P$.

$(\forall P) = X$ iff $P = X$, and $(\forall P) = \emptyset$ otherwise.

Not a really useful definition of knowledge, merely a limit notion.

Having Evidence for a Proposition

E_0 and E are two other global modalities.

Associate to proposition P another proposition E_0P and EP.

$(E_0P) := X$ whenever $\exists e \in E_0$ such that $e \subseteq P$.

$(EP) := X$ whenever $\exists e \in E$ such that $e \subseteq P$.

Having (basic) evidence is by van Benthem and Pacuit. Having (combined) evidence is introduced in the paper.

EP can be interpreted as *having an argument for P*.

Having Factive Evidence for a Proposition

$\Box_0$ and $\Box$ are local modalities.

Associate to proposition P another proposition $\Box_0 P$ and $\Box P$.

$x \in \Box_0 P$:iff $\exists e \in E_0 (x \in e \subseteq P)$.

$x \in \Box P$:iff $\exists e \in E (x \in e \subseteq P)$.

$\Box P$ can be interpreted as having a correct argument for P.

$x \in \Box P$ iff $x \in Int(P)$, so this operator coincides with the interior operator!

Belief à la van Benthem and Pacuit

Global, associates with each proposition P another proposition $BelP$.

$BelP = X$:iff $\bigcap F \subseteq P$ for every $F \in Max_{\subseteq}\mathcal{F}$ ($BelP = \emptyset$ otherwise).

Equivalent to treating Evidence models as Sphere models.

Undesired consequences: we can get $Bel\bot$. *See blackboard.*

Not coherentist :(

(You Better) Belief

Global, associates with each proposition P another proposition BP.

$BP = X$:iff $\forall F \in \mathcal{F}^{finite} \exists F' \in \mathcal{F}^{finite} (F \subseteq F' \wedge \bigcap F' \subseteq P)$.

Read: BP iff P is entailed by all “sufficiently strong” pieces of evidence.

Always consistent!

Also behaves like belief in the standard $KD45$ doxastic logics.

Moreover, it is a purely topological notion!

(You Better) Belief cont'd

Proposition 2. TFAE:

1. BP holds (at any state);
2. every (combined) evidence can be strengthened to some evidence supporting P (I.e. $\forall e \in E \exists e' \in E$ s.t. $e' \subseteq e \cap P$);
3. every argument (for anything) can be strengthened to an argument for P (i.e. $\forall U \in \tau_E - \{\emptyset\} \exists U' \in \tau_E - \{\emptyset\} s.t. U' \subseteq U \cap P$);
4. **there is a justification for P: i.e. some argument for P which is consistent with any available evidence** ($\exists U \in \tau_E$ s.t. $U \subseteq P$ and $U \cap e \neq \emptyset$ for all $e \in E$);
5. P includes some dense open set;
6. $IntP$ is dense in τ_E, i.e. $Cl(IntP) = X$, or equivalently, **X − P is nowhere dense**;
7. $\forall \Diamond \Box P$ holds (at any state: i.e. $\forall \Diamond \Box P \neq \emptyset$).

Conditional Belief

For sets $Q, Q' \subseteq X$ we say that Q' is *Q-consistent* iff $Q \cap Q' \neq \emptyset$.

A body of evidence F is *Q-consistent* iff $\bigcap F \cap Q \neq \emptyset$.

$B^Q P$:iff every finite Q-consistent body of evidence can be strengthened to some finite Q-consistent body of evidence supporting $Q \to P$ ($:= \neg Q \vee P$).

It exists.

Defeasible Knowledge

Local, associates with each proposition P another proposition KP.

$KP := \{x \in X : \exists U \in \tau_E(x \in U \subseteq P \wedge Cl(U) = X)\}$.

KP holds at x iff P includes a dense open neighborhood of x.

Equivalently, $x \in IntP$ and $IntP$ is dense.

K as a knowledge operator

Definition

KP holds at x iff $P \subseteq X$ includes a dense open neighbourhood of x.

Reminder

BP holds iff P includes some dense open set.

A justification for P is a dense open subset of P.

A justification for P is *factive* if it is true in the actual world.

Thus, K-knowledge is correctly justified belief

Note that $x \in KP$ entails that $x \in P$. We have *veracity of knowledge*

Further, $BP = BKP$. Belief is indistinguishable from knowledge.

K is defeasible – knowledge can "get lost".

(Another) defeasibility theory of knowledge

Lehrer: *P* is known if it is believed and there exists a justification for *P* that cannot be defeated by any *true evidence*

This is stronger than *justified true belief* to avoid Gettier-cases:

Example: Sheep in a field

Imagine that you are standing outside a field. You see, within it, what looks exactly like a sheep. What belief instantly occurs to you? Among the many that could have done so, it happens to be the belief that there is a sheep in the field. And in fact you are right, because there is a sheep behind the hill in the middle of the field. You cannot see that sheep, though, and you have no direct evidence of its existence. Moreover, what you are seeing is a dog, disguised as a sheep. Hence, you have a well justified true belief that there is a sheep in the field. But is that belief knowledge?

Quoted from: Internet Encyclopedia of Philosophy: https://iep.utm.edu/gettier/#H4

Critics: This condition is too strong – it excludes cases that we would like to call “knowledge”

Loretta and her taxes

Example: Taxes

Loretta has done her taxes, careful to double check every calculation. Based on this evidence she correctly believes that she owes 500 Dollars.

She asks her accountant to check her tax report. The accountant finds no errors, and so he sends her a reply reading “Your report contains *no* errors”, but he accidentally leaves out the word “no”.

If Loretta would learn the true fact that the accountants reply reads “Your report contains errors”, she would lose her belief that she owes 500 Dollars.

With Lehrer’s definition of knowledge, Loretta thus does not know that she owes 500 Dollars.

Loretta and her taxes – formalized

Recall the example from before (and call it $\mathcal{M}$):

$X = \{x_1, x_2, x_3, x_4, x_5\}$ and $E_0 = \{X, O_1, O_2\}$ where $O_1 = \{x_1, x_2, x_3\}$ and $O_2 = \{x_3, x_4, x_5\}$. Then $E = \{X, O_1, O_2, \{x_3\}\}$. Let x_1 be the actual world.

Generating a topology from E_0 gives us: $\tau_E = \{\emptyset, X, O_1, O_2, \{x_3\}\}$

We find that: $Cl(O_1) = X$ and $x_1 \in Int(O_1) = O_1$. So O_1 is dense and it's an open neighbourhood of x_1. This means that $x_1 \in K(O_1)$. So O_1 is known!

O_1 can be understood as Loretta's direct evidence that she owes 500 Dollars. O_2 can be understood as her evidence that her accountant does not make mistakes in his replies.

Loretta and her taxes – formalized

Consider $\mathcal{M}^{+O_3} = (X, E_0^{+O_3}, V)$, obtained by adding new evidence $O_3 = \{x_1, x_5\}$. Then:

$$E_0^{+O_3} = \{X, O_1, O_2, O_3\} \qquad E^{+O_3} = \{X, O_1, O_2, O_3, \{x_1\}, \{x_3\}, \{x_5\}\}$$

This affects the topology $\tau_{E^{+O_3}}$ generated by E^{+O_3}. In particular: Since $\{x_5\} \in \tau_{E^{+O_3}}$, $X \setminus \{x_5\}$ is closed. As $O_1 \subseteq X \setminus \{x_5\}$, we then get $Cl(O_1) \neq X$.

So O_1 is no longer dense in $\tau_{E^{+O_3}}$!

So by adding the factive evidence O_3 to the model, O_1 is not even believed anymore – there is no longer a justification for O_1.

Remember: O_1 corresponds to Loretta's evidence that she owes 500 Dollars, O_2 to her evidence that the accountant makes no mistakes. O_3 represents the accountant's faulty reply to Loretta.

Misleading defeaters

"The accountant's reply says that Loretta's report contains errors" might be a *true fact*, but it's somehow *misleading*

P. Klein's idea: "A defeater is misleading if it justifies a falsehood in the process of defeating the justification for the target belief."

Misleading evidence

Given a topo-e-model $\mathcal{M}$, a proposition $Q \subseteq X$ is *misleading* at $x \in X$ w.r.t E if there is some $e' \in E^{+Q} \setminus E$ s.t. $x \notin e'$.

So O_3 is misleading at x_1 w.r.t. E: $\{x_5\} \in E^{+O_3} \setminus E$ and $x_1 \notin \{x_5\}$.

Weakening Lehrer's defeasibility theory

Armed with a concept of misleading evidence, we can weaken Lehrer's defeasibility theory of knowledge:

P is known if there exists a justification for P that is undefeated by every non-misleading proposition.

The good news: The knowledge operator K coincides with this:

Equivalence

Let $\mathcal{M}$ be a topo-e-model, and assume $x \in X$ is the actual world. TFAE for all $P \subseteq X$:

1. P is known ($x \in KP$)
2. there is an argument for P that cannot be defeated by any non-misleading proposition; i.e. $\exists U \in \tau_E \setminus \{\emptyset\}$ s.t. $U \subseteq P$ and $U \cap Q \neq \emptyset$ for all non-misleading $Q \subseteq X$.

The Logic

The topological language $\mathcal{L}$ is given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B\varphi \mid K\varphi \mid \forall\varphi \mid B^{\varphi}\varphi| \Box\varphi \mid E\varphi$$

.

I'll highlight a few of the properties of the logics that can be studied using this language.

The semantics is ”obvious”.

A Useful Proposition

Proposition 4 The following equivalences are valid in all topo-e-models.

1. $B\varphi \leftrightarrow \langle K\rangle K\varphi \leftrightarrow \exists K\varphi \leftrightarrow \forall\Diamond\Box\varphi$;
2. $E\varphi \leftrightarrow \exists\Box\varphi$;
3. $E_0\varphi \leftrightarrow \exists\Box_0\varphi$;
4. $K\varphi \leftrightarrow \Box\varphi \wedge B\varphi \leftrightarrow \Box\varphi \wedge \forall\Diamond\Box\varphi$;
5. $B^{\theta}\varphi \leftrightarrow \forall(\theta \rightarrow \Diamond(\theta \wedge \Box(\theta \rightarrow \varphi))$;
6. $\forall\varphi \leftrightarrow B^{\neg\varphi}\bot$.

Theorems

Theorem 1 The system $KD45$ (for the B operator) is sound and complete for $\mathcal{L}_B$.

Theorem 2 The system $S4.2$ (for the K operator) is sound and complete for $\mathcal{L}_K$.

Theorem 3 A sound and complete axiomatization for $\mathcal{L}_{KB}$ is given by Stalnaker's system KB, consisting of the following:

1. The $S4$ axioms and rules for Knowledge K;
2. Consistency of Belief: $B\varphi \to \neg B \neg \varphi$;
3. Knowledge implies Belief: $K\varphi \to B\varphi$;
4. Strong Positive and Negative Introspection for Belief: $B\varphi \to KB\varphi$; $\neg B\varphi \to K\neg B\varphi$;
5. The "Strong Belief" axiom: $B\varphi \to BK\varphi$.

Theorems cont'd

Theorem 4 The following system is sound and complete for $\mathcal{L}_{\forall\Box}$:

1. The S5 axioms and rules for $\forall$;
2. The $S4$ axioms and rules for $\Box$;
3. $\forall\varphi \to \Box\varphi$.

The above one is interesting because all other operators of $\mathcal{L}$ are definable in terms of $\Box, \forall$.

Theorem 5 The following system is sound and complete for $\mathcal{L}_{\forall K}$

1. the S5 axioms and rules for $\forall$;
2. the S4 axioms and rules for K;
3. $\forall\varphi \to K\varphi$;
4. $\exists K\varphi \to \forall\langle K\rangle\varphi$.

Theorems cont'd cont'd

Theorem 6 (Soundness, Completeness, Finite Model Property and Decidability) The logic $\mathcal{L}_{\forall\Box\Box_0}$ is completely axiomatizable and has the fmp, and hence it is decidable. A complete axiomatization is given by the following system:

1. the S5 axioms and rules for $\forall$;
2. The S4 axioms and rules for $\Box$;
3. $\Box_0\varphi \to \Box_0\Box_0\varphi$;
4. Monotonicity for $\Box_0$: from $\varphi \to \psi$, infer $\Box_0\varphi \to \Box_0\psi$;
5. $\forall\varphi \to \Box_0\varphi$;
6. $\Box_0\varphi \to \Box\varphi$;
7. the Pullout Axiom: $(\Box_0\varphi \wedge \forall\psi) \to \Box_0(\varphi \wedge \forall\psi)$.

The End

Questions?