The Intuitionistic Continuum

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February 3, 2023

Contents

Introduction

- Zeno – *Space is discrete and therefore motion is impossible!*
- Anaxagoras – "*Neither is there a smallest part of what is small, but there is always a smaller (for it is impossible that what is should cease to be)*"
- Euclid – *Geometry is the study of the continuous and it is more fundamental. Arithmetic is the study of the discrete.* (But Euclid proved that there are infinite primes by using the length of line segments in an attempt to show the superiority of the continuous!)
- St. Thomas Aquinas – *How many angels can dance on the head of a pin?*
- John Bell (Bell, 2019) has argued that the opposition between the continuous and the discrete has been the biggest problem and the main source of development in the history of mathematics

1.1. Cauchy Sequences

Definition 1.1. Cauchy sequence

A sequence of rationals $\langle x_n \rangle$ is a **Cauchy sequence** iff

$$\forall k \exists n \forall p \, |x_{n+p} - x_n| < 2^{-k}.$$

Definition 1.2. Equivalence classes of Cauchy sequences

$$\langle r_n \rangle \sim \langle s_n \rangle \text{ iff } \forall k \exists n \forall p \, |r_{n+p} - s_{n+p}| < 2^{-k}.$$

Let $\mathcal{C}$ denote the class of Cauchy sequences. Then,

Definition 1.3. The real number line

$$\mathbb{R} := \mathcal{C}/\sim := \{[\langle x_n \rangle_{n \in \mathbb{N}}]_\sim \mid \langle x_n \rangle_{n \in \mathbb{N}} \in \mathcal{C}\}.$$

1.2. Convergence and Completeness of $\mathbb{R}$

Definition 1.4. A Cauchy sequence $\langle x_n \rangle$ *converges* to y

$\lim_{n\to\infty}\langle x_n \rangle = y$ iff $\forall k \exists n \forall p \, |y - x_{n+p}| < 2^{-k}$.

Theorem 1.1.

A real sequence converges iff it is a Cauchy sequence.

This theorem assures us that Cauchy sequences can indeed be identified with real numbers. Under **Cauchy completeness**, real numbers are identified with the limits of these sequences.

1.3. Least Upper Bound Property

A more common (and stronger) method of expressing the completeness of the reals is by showing that $\mathbb{R}$ has

Theorem 1.2. The *least upper bound* property

If S is a nonempty subset of $\mathbb{R}$ that is bounded above, then S has a least upper bound, that is $sup(S)$ exists.

Both these classical approaches for solidifying the completeness of the reals are grounded in points!

Million Dollar Question can we describe the continuum based on discrete points?

As we will see, the intuitionists are not convinced...

2.1. BHK-Interpretation of Intuitionism

- A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B
- **A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B**
- A proof of $A \to B$ is a construction which enables us to transform a proof of A into a proof of B
- $\perp$ has no proof. So a proof of $\neg A$ is a construction which transforms a proof of A into a proof of $\perp$
- **A proof of $\forall x A(x)$ is a construction which tranforms a proof of $d \in D$ into a proof of $A(d)$ (where D is the domain over which the variable x ranges)**
- **A proof of $\exists x A(x)$ is given by *providing* a $d \in D$ along with a proof of $A(d)$**

2.2. Introduction to Bolshevism

Definition 1.5. $\langle r_n \rangle$ is a real number generator iff

$$\forall k \exists n \forall p \, |r_{n+p} - r_n| < 2^{-k}.$$

While this is essentially the classical definition of a Cauchy sequence, the BHK-interpretation of logical connectives ensures that this definition only works when we are given a **(finite) rule or explicit procedure** for finding such an n. Therefore,

- not all Cauchy sequences are intuitionistically acceptable,
- we need to rethink our classical notions of identity and orders on the reals
- The law of trichotomy does not generally hold over the intuitionistic reals!

2.3. A quick note on the Axiom of Choice

A formulation of the Axiom of Choice (AC):

$$\forall\alpha\exists n A(\alpha, n) \rightarrow \exists\varphi\forall\alpha A(\alpha, \varphi(\alpha))$$

- Given the BHK-Interpretation of the quantifiers, this axiom is not problematic.
- Constructivists who dismiss this axiom, dismiss it on the basis of its *classical* interpretation.
- AC is actually used in many intuitionistically acceptable proofs (and we will use it later!)

2.4. Choice Sequences

Warning Things get real weird, real fast...

- A choice sequence α is an ever unfinished process of choosing rational numbers $\alpha(0), \alpha(1), \alpha(2), \ldots$ by the creating subject.
- The creating subject may choose to follow some sort of *rule* in choosing numbers (e.g., she might want to form an ordered sequence of prime numbers where $\alpha = p_1, p_2, p_3, \ldots$) or she may *arbitrarily choose* values for each term.
- Intuitionistically acceptable Cauchy sequences are *lawlike* since we're given an effective procedure for finding n
- At any stage in the creating subject's activity, only a finite number of terms have been chosen.

2.5. Free Choice Sequences

- *Free* choice sequences lead to the idea that any function φ acting on all choice sequences can only determine the value of $\varphi(\alpha)$ for a particular sequence α via a finite initial segment.
- The finite initial segment of a sequence $\langle \alpha(0), \ldots, \alpha(n-1) \rangle$ is denoted as $\overline{\alpha}(n)$.
- Then, if we have a choice sequence β where $\overline{\beta}(n) = \overline{\alpha}(n)$ for some n, we get $\varphi(\beta) = \varphi(\alpha)$, i.e.,

$$\forall \alpha \exists n \forall \beta \in \overline{\alpha}(n)(\varphi(\alpha) = \varphi(\beta)).$$

2.6. Weak Continuity Principle

- Again,
$$\forall\alpha\exists n\forall\beta\in\overline{\alpha}(n)(\varphi(\alpha)=\varphi(\beta)).$$
- Combining this with AC
$$\forall\alpha\exists n\varphi(\alpha,n)\rightarrow\exists\varphi\forall\alpha A(\alpha,\varphi(\alpha)),$$
- We get

The *weak continuity principle* (***WC-N***)

$$\forall\alpha\exists nA(\alpha,n)\rightarrow\forall\alpha\exists n\exists m\forall\beta\in\overline{\alpha}(n)A(\beta,m)$$

- The intuition behind WC-N is that while choice sequences can be wild and unruley things, properties can be ascribed to them based on finite initial segments.
- This is Brouwer's solution to the uncertainty Borel expressed with respect to infinite choice sequences.

2.7. How Classical is WC-N?

Using the intuitionistically acceptable equivalence

$$(A \vee B) \leftrightarrow \exists n[(n = 0 \wedge A) \vee (n \neq 0 \wedge B)],$$

we can reformulate WC-N into its disjunctive counterpart:

WC-N$^\vee$

$$\forall \alpha (A(\alpha) \vee B(\alpha)) \rightarrow$$

$$\forall \alpha \exists n (\forall \beta \in \overline{\alpha}(n) A(\beta) \vee \forall \beta \in \overline{\alpha}(n) B(\beta))$$

While WC-N may seem harmless at first glance, it is actually a radically non-classical principle!

2.8. WC-N Refutes LEM!

Theorem 2.1. The weak continuity principle refutes the law of excluded middle (LEM).

Proof. Since we work within an intuitionistic context, we must first assume the LEM. Specifically, its variation $\forall$−LEM, i.e.,

$$\forall\alpha[\forall x(\alpha(x) = 0) \vee \forall x(\alpha(x) \neq 0)].$$

Then, by WC-N$^{\vee}$ we obtain

$$\forall\alpha\exists n[\forall\beta \in \overline{\alpha}(n)\forall x(\beta(x) = 0) \vee \forall\beta \in \overline{\alpha}(n)\forall x(\beta(x) \neq 0)].$$

If we set α to be the constant function on 0, then for all $\beta \in \overline{\alpha}(n)$

$$\forall x(\beta(x) = 0) \vee \forall x(\beta(x) \neq 0).$$

In this case, both disjuncts fail since when $\beta \notin \overline{\alpha}(n + m)$ for some $m > 0$, then $\forall x(\beta(x) = 0)$ doesn't hold, and when $\beta = \alpha$, $\forall x(\beta(x) \neq 0)$ doesn't hold.

3.1. Trees, Spreads, and Fans

Let $T = (\omega^{<\omega}, \preceq)$ denote a tree where each path is identified with a choice sequence $\alpha, \beta, \gamma, \ldots$ and each node with a finite sequence $\vec{u}, \vec{v}, \vec{w}, \ldots$ where each $\vec{u}$ denotes a class of choice sequences where $\alpha \in \vec{u} \leftrightarrow \exists n(\vec{u} = \overline{\alpha}(n))$.

- We define the partial order $\preceq$ over $\omega^{<\omega}$ as follows:
$$\vec{v} \preceq \vec{u} \leftrightarrow \exists \vec{w}(\vec{v} = \vec{u} * \vec{w}).$$
- A ***tree*** $T = (\omega^{<\omega}, \preceq)$ is a non-empty, rooted, and decidable set of finite sequences. More formally, T is a tree iff
  (*i*) $\varnothing \in T$
  (*ii*) $\forall \vec{u}(\vec{u} \in T \vee \vec{u} \notin T)$
  (*iii*) $\forall \vec{u} \forall \vec{v}(\vec{u} \preceq \vec{v} \wedge \vec{u} \in T \to \vec{v} \in T)$
- A ***spread*** is a tree in which each node has a successor:
  (*iv*) $\forall \vec{u} \in T \exists n(\vec{u} * \langle n \rangle \in T)$
- A ***fan*** is a finitely branching spread satisfying
  (*v*) $\forall \vec{u} \in T \exists k \forall n(\vec{u} * \langle n \rangle \in T \to n \leq k)$
- We say that a sequence α is a ***branch*** of the tree when $\alpha \in T \leftrightarrow \forall n(\overline{\alpha}(n) \in T)$.
- For any decidable predicate A where $\forall \alpha \exists n A(\overline{\alpha}(n))$, we can define a ***bar*** such that $P := \{\vec{u} \in T \mid A(\vec{u}) \wedge \forall \vec{v} \preceq \vec{u} \neg A(\vec{v})\}$.

3.2. Fan Theorem

The ***decidable fan principle*** states that for any fan T, *decidable* property A, and $\alpha \in T$ where $\overline{\alpha}(n)$ satisfies A, then there exists a uniform upper bound k such that $A(\overline{\alpha}(m))$ for some $m \leq k$. Formally,

The decidable fan principle (FAN$_D$)

$$\forall \vec{u}(A(\vec{u}) \vee \neg A\vec{u}) \wedge \forall \alpha \in T \exists n A(\overline{\alpha}(n)) \rightarrow$$
$$\exists k \forall \alpha \in T \exists m(m \leq k \rightarrow A(\overline{\alpha}(m)))$$

FAN$_D$ tells us that, if a decidable property can be found for a branch at some finite stage, then it can be decided for all branches at some finite stage. Strengthening this principle by extending it to properties which are not necessarily decidable, we get

The stronger fan principle (FAN)

$$\forall \alpha \in T \exists n A(\overline{\alpha}(n)) \rightarrow \exists k \forall \alpha \in T \exists m(m \leq k \rightarrow A(\overline{\alpha}(m)))$$

3.3. Compactness

It is interesting at this point to note that **FAN expresses the compactness of a fan**. To see this, we equip onto a fan T a topology of initial segments of sequences in T. This topology is generated by the basis $\mathcal{B} := \{U_{\vec{u}} \mid \vec{u} \in T\}$ where $U_{\vec{u}} := \{\alpha \in T \mid \alpha \in \vec{u}\}$. We let $\{W_i \mid i \in I\}$ be any open cover of T. It follows immediately that

$$\forall \alpha \in T \exists n \exists i \in I (U_{\overline{\alpha}(n)} \subseteq W_i).$$

By FAN, it follows that

$$\exists k \forall \alpha \in T \exists m \leq z \exists i \in I (U_{\overline{\alpha}(m)} \subseteq W_i).$$

Essentially, what we have here is a bar. From this, we can generalize

$$\exists k \forall \alpha \in T \exists i \in I (U_{\overline{\alpha}(k)} \subseteq W_i).$$

But since $\overline{\alpha}(k)$ denotes a finite set of initial segments $\vec{u}_1, \ldots, \vec{u}_n$, we can set each $\vec{u}_p$ to some W_{i_q} where $U_{\vec{u}_p} \subseteq W_{i_q}$. Since k is finite, $\{W_{i_0}, \ldots, W_{i_k}\}$ forms a finite subcover.

Similarly, FAN$_D$ expresses a restricted form of *decidable* compactness – if the basis is specified by a decidable property, then we get compactness.

4. 1. Continuity

- A real-valued function f is *uniformly continuous on an interval* $[u, v]$ iff

$$\forall k \exists m \forall x, y \in [u, v](|x - y| < 2^{-m} \to |f(x) - f(y)| < 2^{-k})$$

- f is *uniformly continuous* iff it is uniformly continuous on every interval.
- A *canonical real number generator* is a way of relating some $x \in [u, v]$ to a particular fan. It is a real number generator $\langle r_n \rangle$ whose generating rule has only finitely many possible choices for each r_n.

Theorem 4.1.

For every closed interval $[u, v]$ we can build a spread T such that every real number $x \in [u, v]$ is generated by a $\langle r_n \rangle \in T$.

- More generally, for every $x \in [0, 1]$, there exists an α such that $x = c_\alpha$.

4.2. Brouwer's Theorem

Theorem 4.2. (Uniform Continuity Theorem) If f is a total function on $[0, 1]$, then f is uniformly continuous on $[0, 1]$.

Proof. Let f be a total function on $[0, 1]$. The interval $[0, 1]$ can be represented by a spread T where

$$f(c_\alpha) = f(x) \text{ for } \alpha \in T.$$

For each n and $x \in [0, 1]$, we can approximate $f(x)$ to within 2^{-n-1}. More formally,

$$\forall \alpha \in T \exists m \, |f(c_\alpha) - m \cdot 2^{-n-1}| < 2^{-n-1}.$$

Since the relationship between α and m is extensional, we apply an 'extended' version of FAN for extensional predicates. By applying FAN, we obtain that for every n, we can find an r such that

$$\forall \alpha \in T \exists m \forall \beta \in \overline{\alpha}(r) \, |f(c_\beta) - m \cdot 2^{-n-1}| < 2^{-n-1}.$$

Now suppose that for $y, z \in [0, 1]$, $|y - z| < 2^{-n-1}$. Then by the above theorem, we can set $y = c_\alpha$ and $z = c_\beta$ such that $\overline{c_\alpha}(r) = \overline{c_\beta}(r)$. We then get

$$\begin{aligned} |f(y) - f(z)| &= |f(c_\alpha) - f(c_\beta)| \\ &\ngtr |f(c_\alpha) - m \cdot 2^{-n-1}| + |f(c_\beta) - m \cdot 2^{-n-1}| \\ &< 2^{-n}. \end{aligned}$$

Since for each n we can effectively find such an r, it follows that f is uniformly continuous on $[0, 1]$.

4.3. Indecomposability of $\mathbb{R}$

Definition 4.1. Indecomposability of a set

A set X is *indecomposable* if $X = A \cup B$ where $A \cap B = \varnothing$ (denoted $X = A + B$ which is called a *proper* decomposition), then $X = A$ or $X = B$.

Theorem 4.3. $\mathbb{R}$ is indecomposable.

If any sets A and B formed a proper decomposition of $\mathbb{R}$, i.e., $\mathbb{R} = A + B$, then in an intuitionistic context this suggests that we have an effective function f which determines for all $x \in \mathbb{R}$

$$f(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in B. \end{cases}$$

This function, however, is clearly not continuous.

4.4. Indecomposability of $\mathbb{R} \setminus \{0\}$ [1/3]

Theorem 4.4. $\mathbb{R} \setminus \{0\}$ is indecomposable.

Proof.
Let $\mathbb{R} \setminus \{0\} = A + B$, and suppose that $A + B$ is a proper decomposition of $\mathbb{R} \setminus \{0\}$, i.e., $A \cap B = \varnothing$ and $A \neq \mathbb{R} \setminus \{0\} \wedge B \neq \mathbb{R} \setminus \{0\}$. So $\forall x \in \mathbb{R}(x \neq 0 \to x \in A \vee x \in B)$.

Now let $\mathbb{R}^r := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}^l := \{x \in \mathbb{R} \mid x \leq 0\}$. Pick $x \in \mathbb{R}^l$ where $x < 0$. We know that $x \in A \vee x \in B$. Suppose $x \in A$. Now assume that $y \in \mathbb{R}^l$ such that $y < 0$ and $y \in B$. Then $(-\infty, 0) = (-\infty, 0) \cap A \cup (-\infty, 0) \cap B$. But then we have a partitioning of $(-\infty, 0)$ into $(-\infty, 0) \cap A$ and $(-\infty, 0) \cap B$. This, however, contradicts the indecomposability of $\mathbb{R}$.

So from $y < 0 \to y \notin B$, we get $y \in B \to y \geq 0$. In other words, $B \subseteq \mathbb{R}^r$. Now let $z > 0$ and suppose $z \in A$. A similar argument shows that $(0, \infty)$ has a proper decomposition which, again, conflicts with Brouwer's theorem. So we similarly get $z \in A \to z \leq 0$, i.e., $A \subseteq \mathbb{R}^l$. We then obtain

$$\forall x \in \mathbb{R}(x \neq 0 \to x \geq 0 \vee x \leq 0).$$

4.4. Indecomposability of $\mathbb{R} \setminus \{0\}$ [2/3]

Theorem 4.4. $\mathbb{R} \setminus \{0\}$ is indecomposable.

In order to formulate our desired sequences for the next part of this proof, we need to make use of Kripke's schema:

$$\exists \alpha \in \{0,1\}^{\omega}(\exists n(\alpha(n) = 1) \leftrightarrow \varphi). \tag{KS}$$

This schema defines binary sequences where the terms of α are equal to 1 when φ holds and 0 otherwise. We can consider a property φ for which there is at most one n such that $\alpha(n) = 1$:

$$\forall n \sum_{x=0}^{n} \alpha(x) \leq 1.$$

Using Kripke's schema, we can generate sequences from the statements $r \in \mathbb{Q}$ and $r \notin \mathbb{Q}$. Let α denote the Kripke-sequence generated from $r \in \mathbb{Q}$, and β that generated by $r \notin \mathbb{Q}$. Formally,

$$\exists n(\alpha(n) = 1) \leftrightarrow r \in \mathbb{Q}$$

and

$$\exists m(\beta(m) = 1) \leftrightarrow r \notin \mathbb{Q}.$$

4.4. Indecomposability of $\mathbb{R} \setminus \{0\}$ [3/3]

Theorem 4.4. $\mathbb{R} \setminus \{0\}$ is indecomposable.

We can then generate another sequence γ from α and β by setting $\gamma(2x) = \alpha(x)$ and $\gamma(2x+1) = \beta(x)$. We the define the function

$$c_n := \begin{cases} (-2)^{-n} & \text{if } \forall k \leq n(\gamma(k) = 0) \\ (-2)^{-k} & \text{if } k \leq n \wedge \gamma(k) = 1. \end{cases}$$

In the case where 1 does not appear in the sequence, then $c = (c_n)_n = 0 \leftrightarrow \forall k(\gamma(k) = 0) \leftrightarrow r \notin \mathbb{Q} \wedge \neg r \notin \mathbb{Q}$ which is a contradiction. So $c \neq 0$. So we get that $c \geq 0 \vee c \leq 0$. By construction, it follows that

$$c \geq 0 \leftrightarrow \neg \exists n(\beta(n) = 1) \leftrightarrow \neg r \notin \mathbb{Q}$$

and

$$c \leq 0 \leftrightarrow \neg \exists n(\alpha(n) = 1) \leftrightarrow r \notin \mathbb{Q}.$$

Since r is arbitrary, we have $\forall r \in \mathbb{R}(r \notin \mathbb{Q} \vee \neg r \notin \mathbb{Q})$. This, however, is refuted by the indecomposability of $\mathbb{R}$. So the decomposition of $\mathbb{R} \setminus \{0\}$ into $A + B$ could not have been proper.

4.5. $\mathbb{R} \setminus \mathbb{Q}$ is Indecomposable

Theorem 4.5.

$\mathbb{R} \setminus \mathbb{Q}$ is indecomposable

"The indecomposability of $\mathbb{R}$ is a peculiar feature of constructive universa, it shows that $\mathbb{R}$ is much more closely knit in constructive mathematics, than in classically mathematics. The classically comparable fact is the topological connectedness of $\mathbb{R}$. In a way this characterizes the position of $\mathbb{R}$: the only (classically) connected subsets of $\mathbb{R}$ are the various kinds of segments. **In intuitionistic mathematics the situation is different; the continuum has, as it were, a syrupy nature, one cannot simply take away one point. In the classical continuum one can, thanks to the principle of the excluded third, do so. To put it picturesquely, the classical continuum is the frozen intuitionistic continuum. If one removes one point from the intuitionistic continuum, there still are all those points for which it is unknown whether or not they belong to the remaining part"** (van Dalen, 1997)

4.6. Philosophical Intuitions on the Continuum

"In traditional analysis, the continuum appeared as a set of its points; it was considered merely as a special case of the basic logical relationship of *element and set* [...] The fact, however, *that it has parts*, is a fundamental property of the continuum; and so (in harmony with intuition, so drastically offended against by todays 'atomism') this relationship is taken as the basis for the mathematical treatment of the continuum in Brouwer's theory." (Weyl, 1921)

"This is the real reason why the method used in delimiting subcontinua and in forming continuous functions starts out from *intervals* and not *points* as the primary elements of construction. Admittedly, a set also has parts. Yet what distinguishes the parts of sets in the realm of the 'divisible' is the existence of 'elements' in the set-theoretical sense, that is, that the existence of *parts that themselves do not contain any further parts* [...] In contrast, it is part of the nature of the continuum that *every part of it can be further divided without limitation*." (Weyl, 1921)

What does this mean for Topology?

- How do these ideas of the continuum extend into topology?

5.1. What does this mean for Topology?

- How do these ideas of the continuum extend into topology?

Pointless Topology!

5.2. Refresher on Frames & Locales [1/2]

Definition 5.1. A Lattice L is a *complete lattice* iff

every subset has a supremum (l.u.b.) and an infimum (g.l.b.).

Let L and L' be lattices and $f : L \to L'$.

Definition 5.2. f is a *lattice homomorphism* iff for any $a, b \in L$,

$$f(a \vee b) = f(a) \vee f(b)$$
$$f(a \wedge b) = f(a) \wedge f(b)$$

Let X be a topological space. We call $\Omega(X)$ a *frame* when it satisfies:

(1) For any $x_i \in \Omega(X)$ where $i \in I$, their join $\bigvee x_i \in \Omega(X)$.

(2) For all $b \in \Omega(X)$ and $A \subseteq \Omega(X)$,

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\}.$$

5.2. Refresher on Frames & Locales [2/2]

- $\mathbf{Frm}^{op} = \mathbf{Loc}$
- $\Omega :$ **Top** $\rightarrow$ **Loc**. We can go from topological spaces to the 'lattice of opens'
- But to go from **Loc** to **Top** we need to define points *in terms of opens* (this is what Brouwer does!)

5.3. Points

Definition 5.3. A filter F is *completely prime* iff

for all collections $(U_i)_{i \in I}$ of open sets such that $\bigcup_{i \in I} U_i \in F$, we have that $U_i \in F$ for some $i \in I$.

Definition 5.4. A topological space (X, τ) is called *sober* if

each completely prime filter is the neighbourhood filter, $\mathcal{N}(x)$, of a unique $x \in X$.

- A *point* in a locale L is a completely prime filter $F \subseteq L$.
- We can get points out of *neighbourhoods*!
- "To represent the continuous connection of the points, traditional analysis, given its shattering of the continuum into a set of isolated points, had to have recourse to the concept of a *neighbourhood*." (Weyl, 1921)

5.4. Spatial/Non-Spatial Locales

- A locale L is said to be *spatial* if it is isomorphic to a $\Omega(X)$.
- A locale L is said to *have enough points* if, given any two opens $U, V \in L$, $U = V$ if precisely the same points of L belong to U as belong to V.
- But we can have non-spatial Locales!
- What does point-set topology have to say about these?
- *"The question whether a given locale is spatial is intimately related with the existence of appropriate ideal objects whose existence typically hinges on the axiom of choice or one of its variants."* (Blechschmidt, 2020)

5.5. What is $\mathbb{R}$?

- It is consistent with intuitionism that the reals are spatial and also that they are not spatial.
- Since equality is not generally decidable in intuitionism, given a point x, the equality $\mathbb{R} = \mathbb{R} \setminus \{x\} \cup \{x\}$ is not always decidable.
- The van Dalen result holds only in the case of spatial reals.
- (Fourman & Hyland, 1979) showed that the compactness of 2^ω and the local compactness of $\mathbb{R}^D$ can fail in non-spatial locales.
- While in classical analysis $\mathbb{R}^C \cong \mathbb{R}^D$, in non-spatial locales $\mathbb{R}^C \subseteq \mathbb{R}^D$.

5.5. What is $\mathbb{R}$?

- What are the decomposability properties of $\mathbb{R}$ in non-spatial locales?
- Are the discrete cuts problematic? or is recomposition the issue?
- The theory of locales provide us with a way of recovering point-set topology from open sets (at least sober spaces which contain all Hausdorff spaces!).
- We can also continue to do topology in contexts where we don't have choice principles.
- Intuitionism seems to provide us with a rich and fine-grained theory of the continuum.

"*It is nonsense to regard the continuum as a finished being*"
– Weyl, 1921