Shower of Semantics

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Outline

1. *d*-Semantics
   - c-Semantics and Definability
   - Derived Set Semantics
   - *d*-Definability
     - K4 and T_D-Spaces
     - GL and Scattered Spaces
2. Intuitionistic logic as a logic of space

Topological Semantics (c-semantics)

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Recap from guest lecture:

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  $\mathcal{M}, x \models \Diamond\varphi$ iff $\forall U \in \tau$ such that $x \in U$, $\exists y \in U$ with $\mathcal{M}, y \models \varphi$

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  $\mathcal{M}, x \models \Diamond\varphi$ iff $\forall U \in \tau$ such that $x \in U$, $\exists y \in U$ with $\mathcal{M}, y \models \varphi$
- Essentially like any other modal logic, we have seen:
  - topo-bisimulation
  - "topo-p-morphisms" (interior maps and open subspaces)
  - "topo-disjoint union" (topological sums)

Topological Semantics

Theorem

(McKinsey and Tarski, 1944)

- *S4 is complete wrt all topological spaces*
- *S4 is complete wrt any dense-in-itself metrizable space*
- *S4 is complete wrt the real line* $\mathbb{R}$
- *S4 is complete wrt the rationals* $\mathbb{Q}$

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- every closed subset is also open (S5)
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Undefinabilities in c-semantics

- separation axioms (T_0, ..., T_4)
- compactness and connectedness
- dense-in-itself

Derived Set Semantics

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Truth of a modal formula in a state w of a model $\mathcal{M}$ is defined by

- $\mathcal{M}, w \models_d p$ iff $w \in \nu(p)$
- $\mathcal{M}, w \models_d \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
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- $\mathcal{M}, w \models_d \neg\varphi$ iff $\mathcal{M}, w \not\models_d \varphi$
- $\mathcal{M}, w \models_d \Diamond\varphi$ iff $\forall U \in \tau(w \in U \to \exists v \in U - \{w\} : \mathcal{M}, v \models_d \varphi)$

Derived Set Semantics and wK4

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This gives us the following axioms for d-semantics:

- $\Diamond(p \vee q) \equiv \Diamond p \vee \Diamond q$ (K)
- $\Diamond\Diamond p \rightarrow p \vee \Diamond p$ (w4)

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- $\lozenge(p \vee q) \equiv \lozenge p \vee \lozenge q$ (K)
- $\lozenge\lozenge p \to p \vee \lozenge p$ (w4)

The logic K+w4 is called **weak K4** or **wK4**. It follows that wK4 is sound wrt d-semantics.

Derived Set Semantics and wK4

Theorem (Esakia, 2001)

The modal logic wK4 is sound and complete wrt all topological spaces.

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To prove this we first give some notation:
We denote the reflexive closure (resp. irreflexive fragment) of a frame $\mathfrak{F} = (\mathcal{W}, R)$ as $\overline{\mathfrak{F}} = (\mathcal{W}, \overline{R})$ (resp. $\underline{\mathfrak{F}} = (\mathcal{W}, \underline{R})$).

Derived Set Semantics and wK4

Lemma

Let $\mathfrak{F} = (X, R)$ *be a wK4-frame and* $A \subseteq X$. *In* $\overline{\mathfrak{F}}$ *we have*
$d(A) = \underline{R}^{-1}(A)$.
(Whereas $d(A)$ *is defined in terms of* $\overline{\mathfrak{F}}$ *being an Alexandroff space.)*

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Proof of the theorem: See blackboard.

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d-Definability

Definability

Topo-definability

We say that a class K of topological spaces is **topologically definable** or simply **topo-definable** if there exists a set of modal formulas Γ such that for each topological space $\mathcal{X}$ we have $\mathcal{X} \in K$ iff $\mathcal{X} \vDash_c \Gamma$.

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Definability

Expressive power

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- Topo-definability results will automatically transfer into *d*-definability results.
- There are *d*-definable topological properties that are not topo-definable.

K4 and T_D-Spaces

T_D-spaces

Definition

A topological space $\mathcal{X}$ is said to satisfy the T_D-**separation axiom** or is simply T_D if for every point $x \in \mathcal{X}$, there exist an open U and closed F such that $U \cap F = \{x\}$.

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- Every T_1 space is a T_D space.
- Every T_D space is a T_0 space.

Property of T_D-spaces

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Proof.

($\Rightarrow$) Suppose $x \notin d(A)$.
Then there is an open neighbourhood U of x such that $U \setminus \{x\} \cap A = \emptyset$.
By T_D there are open V and closed F such that $\{x\} = V \cap F$. Then $U \cap V$ is still an open neighbourhood of x.
We show that $(U \cap V) \cap d(A) = \emptyset$:

> Assume there is $y \in (U \cap V) \cap d(A)$. Then $y \notin F$, as $V \cap F = \{x\}$ and $y \neq x$. So $(U \cap V) \setminus F$ is an open neighbourhood of y that has empty intersection with A, which contradicts that $y \in d(A)$.

So $x \notin Cl(d(A))$. As $d(A) \subseteq Cl(A)$, we obtain that $x \notin dd(A)$. □

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Theorem (4 axiom d-defines the class of T_D-spaces.)

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Theorem

K4 *is sound and complete wrt T_D-spaces.*

T_0 spaces

Definition

Let $(\mathcal{X}, \tau)$ be a topological space. We say that two points x, y are **topologically distinguishable** if there exists an open neighbourhood $U_{x,y}$ such that either $x \in U_{x,y}$ and $y \notin U_{x,y}$ or $y \in U_{x,y}$ and $x \notin U_{x,y}$. We say that the space $\mathcal{X}$ is T_0 if all pairs of points are topologically distinguishable.

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Let $t_0 = p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q)$.

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Let $\mathrm{wK4T_0} := \mathrm{wK4} + t_0$.

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Theorem

$\mathrm{wK4T_0}$ *is sound and complete wrt T_0-spaces.*

GL and Scattered Spaces

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The class of scattered spaces is not topo-definable in c-semantics.

Examples of scattered spaces

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   - Let $\mathbb{R}$ be the real line equipped with the Euclidean topology τ.
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   - Let $\mathbb{I} = \mathbb{R} - \mathbb{Q}$ be the set of irrational numbers, and let τ'' be the subspace topology on $\mathbb{I}$ of $\mathbb{R}$ under τ':
   - Then $(\mathbb{I}, \tau'')$ is scattered, since every point in it is isolated.

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Proof.

($\Leftarrow$)

Let $A \subseteq \mathcal{X}$ be nonempty. We show that A has an isolated point.

- If $d(A)$ is empty, we are done.
- Otherwise, take any $x \in d(A)$, so $x \in d(A \setminus d(A))$.
  Since x is a limit of isolated points of A, there must be at least one such point.

□

Proof continues

(⇒)

Suppose $\mathcal{X}$ is scattered, $A \subseteq \mathcal{X}$ and $x \in d(A)$.
Consider any open neighborhood U of x. Since $U \cap A$ is nonempty, it has an isolated point y.

- If $y = x$, this contradicts with $x \in d(A)$. Suppose x is isolated in $U \cap A$. Then there is an open neighbourhood V of x and $V \cap (U \cap A) = \{x\}$. But $V \cap (U \cap A) = (V \cap U) \cap A$ and $V \cap U$ is also an open neighbourhood of x, which leads to a contradiction.
- If $y \neq x$, then there is a open neighbourhood J of y and $J \cap (U \cap A) = \{y\}$. Since $J \cap (U \cap A) = (J \cap U) \cap A$ and $J \cap U$ is also an open neighbourhood of y, y is an isolated point of A, that is, $y \in A \setminus d(A)$.

Hence, $x \in d(A \setminus d(A))$. The inclusion $d(A \setminus d(A)) \subseteq d(A)$ follows from the monotonicity of d. Therefore, $d(A) = d(A \setminus d(A))$ holds.

GL and scattered spaces

Lemma

For any space $\mathcal{X}$, $d(A) = d(A \setminus d(A))$ *for every* $A \subseteq \mathcal{X}$ *iff* $\mathcal{X} \vDash_d \Box(\Box p \to p) \to \Box p$.

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Theorem (Esakia, Simmons, Löb d-defines the class of scattered spaces)

A space $\mathcal{X}$ *is scattered iff* $\mathcal{X} \vDash_d$ *Löb.*

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The Gödel-Löb provability logic GL is obtained by adding to K the Löb formula $\Box(\Box p \to p) \to \Box p$.

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Theorem (Esakia, 1981)

GL *is sound and complete wrt scattered spaces.*

Intuitionistic Propositional Calculus (IPC)

Definition (Language $\mathcal{L}_{int}$)

$$\mathcal{L}_{int} := \{\wedge, \vee, \rightarrow, \bot\}$$

Intuitionistic Propositional Calculus

Ax-1 $\varphi \rightarrow (\psi \rightarrow \varphi)$

Ax-2 $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)$

Ax-3 $\varphi \wedge \psi \rightarrow \varphi$

Ax-4 $\varphi \wedge \psi \rightarrow \psi$

Ax-5 $\varphi \rightarrow \varphi \vee \psi$

Ax-6 $\psi \rightarrow \varphi \vee \psi$

Ax-7 $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)$

Ax-8 $\bot \rightarrow \varphi$

BHK-semantics

Informal clauses

- A proof of $\varphi \wedge \psi$ consists of a proof of φ and a proof of ψ;
- A proof of $\varphi \vee \psi$ consists of a proof of φ or a proof of ψ;
- A proof of $\varphi \to \psi$ consists of a method which turns a proof of φ into a proof of ψ;
- A proof of $\neg\varphi$ consists of a method which turns a proof of φ into a proof of $\bot$;
- $\bot$ has no proof.

IPC vs CPC

Famously, some classical theorems are not intuitionistically valid:

Law of excluded middle (LEM)

$\nvdash_{IPC} \varphi \vee \neg\varphi$

Peirce's law

$\nvdash_{IPC} ((\varphi \to \psi) \to \varphi) \to \varphi)$

Double negation elimination (DNE)

$\nvdash_{IPC} \neg\neg\varphi \to \varphi$

Topological semantics

Definition (Topological model)

A **topological model** is a triple $\mathcal{T} = (X, \tau, v)$, where (X, τ) is a topological space, and $v : \mathrm{Prop} \to \tau$.

Definition (Truth-set)

Let $\mathcal{T}$ be a topological model, and α, β be arbitrary formulas. Then:

- $p_{\mathcal{T}} = v(p)$
- $\bot_{\mathcal{T}} = \emptyset$
- $\alpha \wedge \beta_{\mathcal{T}} = \alpha_{\mathcal{T}} \cap \beta_{\mathcal{T}}$
- $\alpha \vee \beta_{\mathcal{T}} = \alpha_{\mathcal{T}} \cup \beta_{\mathcal{T}}$
- $\alpha \to \beta_{\mathcal{T}} = Int(\alpha^c_{\mathcal{T}} \cup \beta_{\mathcal{T}})$
- $\neg\alpha_{\mathcal{T}} = Int(\alpha^c_{\mathcal{T}})$

Countermodel to LEM

Countermodel

Take $X = \mathbb{R}$. Set $v(p) = \mathbb{R}^+$.

Proof.

Then $\neg p = \mathbb{R}^-$. But $p \vee \neg p = \mathbb{R}^+ \cup \mathbb{R}^- = \mathbb{R} \setminus \{0\} \neq \mathbb{R}$. □

Countermodel to Peirce's Law

Countermodel

Take $X = \mathbb{R}$. Set $v(p) = \mathbb{R} \setminus \{0\}$ and $v(q) = \emptyset$.

Proof.

$p \to q = Int(p^c \cup q) = Int(\{0\} \cup \emptyset) = \emptyset.$
$(p \to q) \to p) = Int(\mathbb{R} \cup \mathbb{R}) = \mathbb{R}.$
$((p \to q) \to p) \to p) = Int(\emptyset \cup (\mathbb{R} \setminus \{0\}) = \mathbb{R} \setminus \{0\} \neq \mathbb{R}.$ □

Countermodel to DNE

Countermodel

Take $X = \{0, 1\}$, with $\tau = \{\emptyset, X, \{0\}\}$. Set $v(p) = \{0\}$.

Proof.

$\neg\neg p = X$.

$\neg\neg p \to p = Int(\emptyset \cup \{0\}) = \{0\} \neq X$. □

Heyting Algebra

Definition (Heyting algebra)

An **Heyting algebra** $\mathfrak{A}$ is an algebraic structure $(A, \wedge, \vee, \rightarrow, 0, 1)$ such that:

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
- The $\rightarrow$ operation is defined as follows:

$$x \rightarrow x = 1$$

$$x \wedge (x \rightarrow y) = x \wedge y$$

$$(x \rightarrow y) \wedge y = y$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$$

- $\neg a := a \rightarrow 0.$

Examples of Heyting algebras

Example 1

Every chain $\mathfrak{C}$ with a least and a greatest element is a Heyting algebra satisfying:

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b. \end{cases}$$

Example 2

Consider (X, τ) topological space. An algebraic structure $(\tau, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra, with:

$$U \to V := Int(U^c \cup V)$$

for $U, V \in \tau$.

Examples of Heyting algebras

Example 3

Every Boolean algebra $\mathfrak{B}$ is a Heyting algebra, where we have:

$$a \to b = \neg a \vee b$$

for $a, b \in B$.

Proposition (N. Bezhanishvili, de Jongh)

Let $\mathfrak{A}$ be a Heyting algebra. The following are equivalent:

- $\mathfrak{A}$ is a Boolean algebra;
- $a \vee \neg a = 1$ for all $a \in A$;
- $\neg\neg a = a$ for all $a \in A$.

Algebraic semantics

Definition (Algebraic model for IPC)

Let $\mathfrak{A}$ be a Heyting algebra. Then $\mathcal{A} = (\mathfrak{A}, v)$ is an algebraic model for IPC, where the valuation function $v : \mathrm{Prop} \to A$ is defined as follows:

- $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$
- $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$
- $v(\varphi \to \psi) = v(\varphi) \to v(\psi)$
- $v(\bot) = 0$

Definition (Validity)

A formula φ is **valid** in an algebra $\mathfrak{A}$ (written $\mathfrak{A} \models \varphi$) iff, for every valuation v on $\mathfrak{A}$, $v(\varphi) = 1$.

Soundness and completeness

Theorem (Algebraic soundness)

If $\vdash_{IPC} \varphi$, then $\mathfrak{A} \models \varphi$, for all $\mathfrak{A} \in HA$.

Theorem (Algebraic completeness (Jaśkowski 1936, Tarski-McKinsey 1946))

IPC is complete with respect to finite Heyting algebras, that is, if $\mathfrak{A} \models \varphi$ then $\vdash_{IPC} \varphi$, for $\mathfrak{A}$ finite Heyting algebra.