Stone Duality

A bridge between topology and algebra

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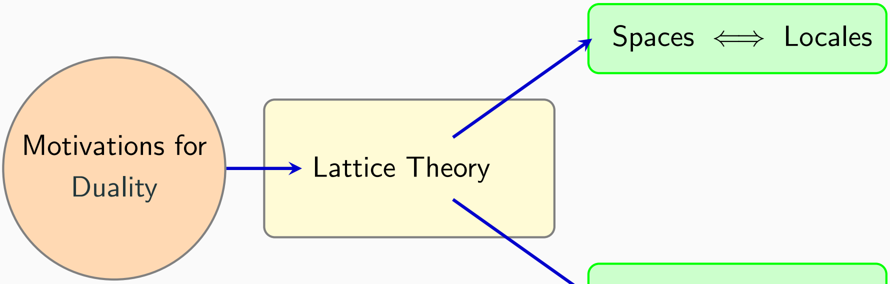
Overview



Motivations for Duality
Lattice Theory
Spaces $\Longleftrightarrow$ Locales
Spaces $\Longleftrightarrow$ Algebras

Motivation

Motivation

Semantics paradigms

- Algebraic semantics
- Topological semantics

Example: Intuitionistic logic

- Algebraic semantics: Heyting algebras
- Topological semantics: S4 Kripke frames

First steps

Basics in both fields

- Topology
- Lattice theory

Basic Lattice Theory

Partial orders

Partial Order

Tuple $\langle X, R \rangle$ such that

1. R reflexive
2. R transitive
3. R antisymmetric

Lattices

Lattice

Tuple $\langle X, R, \wedge, \vee \rangle$ such that

1. $\langle X, R \rangle$ a *partial order*
2. $x \wedge y = y \wedge x$ (Commutativity)
3. $x \vee (y \vee z) = (x \vee y) \vee z$ (Associativity)
4. $x \wedge x = x$ (Idempotence)
5. $x \vee (y \wedge x) = x$ (Absorption)

Lattice homomorphisms

Structure-preserving map between lattices (categorically analogous to continuous maps between topological spaces).

Lattice homomorphism

Map $f : L \to L'$ such that

1. $f(x \wedge y) = f(x) \wedge f(y)$
2. $f(x \vee y) = f(x) \vee f(y)$

Important lattice sub-types

Distributive lattices

Lattice $\langle X, R, \wedge, \vee \rangle$ such that

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Complete lattices

Lattice $\langle X, R, \wedge, \vee \rangle$ such that

1. $M \subseteq L \implies \bigwedge M \in L$
2. $M \subseteq L \implies \bigvee M \in L$

where L is our lattice.

Boolean algebras

- Lattice with additional structure
  - Complement
  - Top
  - Bottom
- Complete semantics for classical logic
  - Complement $\rightarrow$ "Not"
  - Top $\rightarrow$ "True"
  - Bottom $\rightarrow$ "False"

Boolean algebras (cont.)

Boolean algebra

Tuple $\langle A, \wedge, \vee, ', 0, 1\rangle$ such that

1. $\langle A, \wedge, \vee\rangle$ a *distributive lattice*
2. $x \wedge 0 \approx 0, x \vee 1 \approx 1$ (Identities)
3. $x \wedge x' \approx 0, x \vee x' \approx 1$ (Complements)

Boolean algebras (cont.)

Example: Power sets

Claim: Given set X, $\mathcal{P}(X)$ forms a Boolean algebra $\langle \mathcal{P}(X), \cap, \cup, \overline{\cdot}, \varnothing, X \rangle$.

Proof: Where U, V, W are arbitrary subsets of X (and, therefore, members of $\mathcal{P}(X)$) ...

1. $\langle \mathcal{P}(X), \cap, \cup, \rangle$ is a distributive lattice
    1.1 $\langle \mathcal{P}(X), \subseteq \rangle$ is a partial order
    1.2 $U \cap V = V \cap U$ (Commutativity)
    1.3 $U \cup (V \cup W) = (U \cup V) \cup W$ (Associativity)
    1.4 $U \cap U = U$ (Idempotence)
    1.5 $U \cup (V \cap X)$ (Absorption)
    1.6 $U \cup (V \cap W) = (U \cup V) \cap (U \cup W)$ (Distributivity)
2. $U \cap \varnothing = \varnothing$ (Identities)
3. $U \cap \overline{U} = \varnothing$ (Complements)

Filters & ideals

Given a Boolean algebra A ...

Filter

Subset $F \subseteq A$ such that

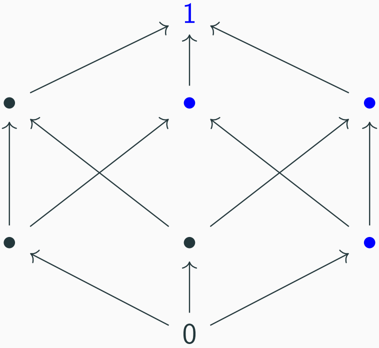
1. $1 \in F$
2. $x, y \in F \implies x \wedge y \in F$
3. $x \in F$ & $xRy \implies y \in F$

Ideal

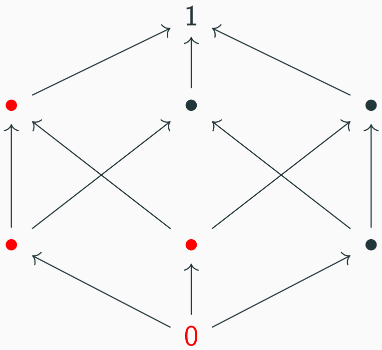
Subset $I \subseteq A$ such that

1. $0 \in I$
2. $x, y \in I \implies x \vee y \in I$
3. $y \in I$ & $xRy \implies x \in I$

Example Filter



Example Ideal



Ultrafilters & maximal ideals

Ultrafilter

Filter F that is maximal with respect to the property that $0 \notin F$.

Maximal Ideal

Ideal I that is maximal with respect to the property that $1 \notin F$.

Note that 'ultrafilters' also go by the names 'maximal filters' and 'fluffy filters'.

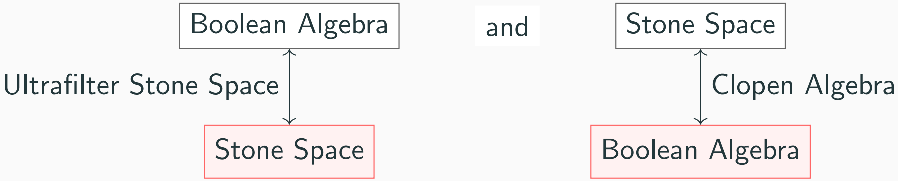
Boolean Representation Theorem

The Theorem

Our Goal
Provide a means of changing between Boolean Algebras and certain Topological Spaces.

Stone Space
A Topological Space which is Hausdorff, Compact, and has a basis of clopens is a **Stone Space.**

Proof Structure



Algebra Space

Our Construction
Given a Boolean Algebra $\mathbf{B}$, we define $\mathbf{B}^*$ to be the set of ultrafilters on $\mathbf{B}$ with the topology generated by

$$N_a := \{U \in \mathbf{B}^* \mid a \in U\}$$

for each $a \in \mathbf{B}$

Basis of Clopens

Proof

We know	$\forall U\ a \vee \overline{a} = 1 \in U,$
so	$a \in U$ or $\overline{a} \in U$
which means	$U \notin N_a \rightarrow a \notin U$
	$\rightarrow \overline{a} \in U$
	$\rightarrow U \in N_{\overline{a}}$

Thus, $\overline{N_a} = N_{\overline{a}}$ and our basis is a basis of clopens.

Hausdorff

Proof
Let $U_1 \neq U_2$. Pick $a \in U_1 - U_2$. Thus, $a \in U_1$ and $\overline{a} \in U_2$ so $N_a \cap N_{\overline{a}} = \emptyset$, $U_1 \in N_a$, and $U_2 \in N_{\overline{a}}$.

Compactness

Proof
Let $(N_a)_{a \in J}$ for $a \in J \subseteq \mathbf{B}$ be a cover of $\mathbf{B}^*$. First, we assume that there exists a finite $J_0 \subseteq J$ such that $\bigvee J_0 = 1$.

Compactness

Proof

$$
\begin{aligned}
&\text{Since} && \bigvee J_0 = 1 \\
&\text{we know} && \forall U \ \bigvee J_0 \in U \\
&\text{or} && \forall U \ a_1 \vee a_2 \ldots \vee a_n \in U \\
&\text{by maximality} && \exists a_i \in U \\
&\text{so} && U \in N_{a_i}.
\end{aligned}
$$

Compactness

Proof

Now, let us assume that no finite subset of J has 1 in its join.

Thus, $J \subset M$ where M is a maximal ideal.

Compactness

Proof

Let	$U = \overline{M}$
since	$J \subset M$
we know	$J \cap U = \emptyset$
so	$\forall a \in J \; a \notin U$
which means	$U \notin (N_a)_{a \in J}$.

$\lightning$

Full Proof

Let $(N_a)_{a \in J}$ for $a \in J \subseteq \mathbf{B}$ be a cover of $\mathbf{B}^*$. First, let us assume that for all finite $J_0 \subseteq J$, $\bigvee J_0 \neq 1$. Thus, $J \subseteq M$ where M is a maximal ideal. Let F be an ultrafilter such that $F = \overline{M}$. Since $J \subseteq M$, we know that $F \cap J = \emptyset$. Thus, for all $a \in J$, $F \notin N_a$. This contradicts the fact that $(N_a)_{a \in J}$ is a cover. Thus, there must exist a finite $J_0 \subseteq J$ such that $\bigvee J_0 = 1$.

Furthermore, this means that for all ultrafilters U, $\bigvee J_0 \in U$. Thus, since ultrafilters are maximal, for all $U \in \mathbf{B}^*$, $\exists a \in J_0$ such that $a \in U$ and thus $U \in N_a$. This means that $(N_a)_{a \in J_0}$ is a finite cover of $\mathbf{B}^*$.

Space to Algebra

Stone Space to Boolean Algebra

If X is a Stone Space, take the sub-Algebra of the Powerset Algebra that contains only the clopen subsets of X. Call this Boolean Algebra X^*.

Proving the Duality

1. We want to show that $\mathbf{B} \cong \mathbf{B}^{**}$
2. and $X^{**} \cong X$.

$\mathbf{B} \cong \mathbf{B}^{**}$

Isomorphism
Define $f : a \to N_a$. We want to show this is an isomorphism.

Homomorphism

$$\begin{aligned} U \in N_a \cup N_b &\iff U \in N_a \text{ or } U \in N_b \\ &\iff a \in U \text{ or } b \in U \\ &\iff a \vee b \in U \\ &\iff U \in N_{a \vee b} \end{aligned}$$

Injectivity

Proof
Let $a \neq b$.

$$
\begin{aligned}
&\text{Thus} && (a \wedge b) \vee \overline{(a \vee b)} \neq 0 \\
&\text{so} && (a \wedge b) \vee \overline{(a \vee b)} \notin M. \\
&\text{Let} && U = \overline{M} \\
&\text{by closure} && a \wedge b \in U \text{ and} \overline{(a \vee b)} \in U \\
&\text{so} && N_a \neq N_b.
\end{aligned}
$$

Surjectivity

Proof
Let N be clopen in $\mathbf{B}^*$. Thus N has an open cover (N is open), and N is compact (N is closed). Thus, $N = N_{a_1 \vee a_2 \ldots \vee a_n} = N_b$.

Isomorphism

Homomorphism

$+$

Injectivity

$+$

Surjectivity

$=$

Isomorphism

$X \cong X^{**}$

Homeomorphism
Define $f : x \to \{N \in X^* \mid x \in N\}$.

Injectivity
$f(x)$ is an ultrafilter on X^* and from X being Hausdorff, f is injective.

Surjectivity

Proof

X^{**} is a topology of ultrafilters on X^*. Take $U \in X^*$. This has the finite intersection property, so by compactness $\bigcap U \neq \emptyset$. Take $x \in \bigcap U$, we know that $U \subset f(x)$, so $U = f(x)$ by maximality.

Homeomorphism

Clopens in X^{**}
Using our definitions: $\{U \mid N \in U\}$ where N is clopen in X.

Open
$f(N) = \{U \mid \exists x \in N \; f(x) = U\}$ which mean $f(N) = \{U \mid N \in U\}$ (since N is clopen).

Continuous
$f^{-1}\{U \mid N \in U\} = \{x \in X \mid f(x) = U\} = \{x \in X \mid x \in N\} = N.$

Summary

1. Powerset Algebra provides an intuition about algebraic spaces.
2. The set of clopens forms an algebra and an ultrafilter.
3. Our "stone" operation is a duality/equivalence.

Locale Theory

Bridges in mathematics

Static vs Dynamic unification (Caramello)

Algebras

Boolean Algebras **Generalisation** *Heyting Algebras*

Locales

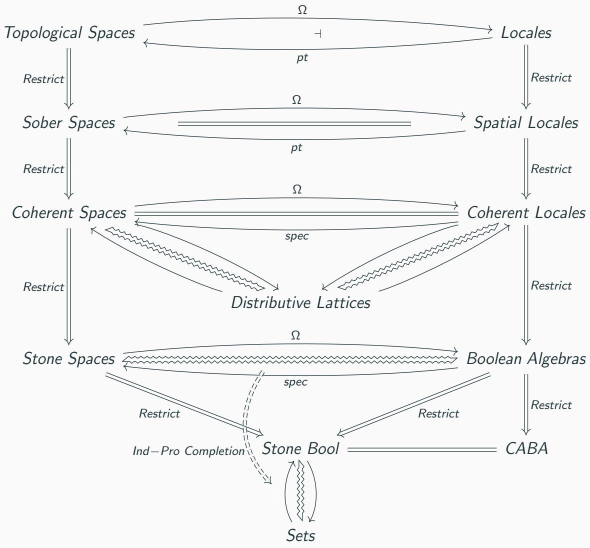
Topological Spaces **Bridges** *Lattices*

Bridges in mathematics (cont.)

- The Boolean Representation Theorem is hinting at a bridge
- What is this bridge?
- Can we obtain interesting results?

- A warning: we will *not* get to the **BRT** in this talk
- Reason: things are slightly complicated

The full picture



Today

We will cover the beginning

The rest is left as an exercise

Concepts in category theory

Category

A collection of mathematical objects of the same type paired with the collection of structure-preserving maps between them.

Examples:

- (Groups, group homomorphisms)
- (Sets, functions)
- (Posets, monotone maps)

Dual

Given a category $\mathcal{C}$, the *dual* of $\mathcal{C}$ (denoted $\mathcal{C}^{op}$ for 'opposite'), is $\mathcal{C}$ with all of the arrows flipped.

Note: *These are not set-theoretic function inverses!*

Equivalence

Two categories are *equivalent* if they are indistinguishable, modulo presentation.

Note: *If they are as in the proof of the* **BRT**.

From spaces to locales: Spaces

- Topological space $\mathbf{X} = (X, \tau)$
  - $\mathbf{X} \mapsto X$ "Forget"
  - $\mathbf{X} \mapsto \tau$ "Lattice of opens"
    - Do we keep "enough" topological data?
- **Top**
  - Category of topological spaces
  - Objects: topological spaces
  - Maps: continuous maps

From spaces to locales: Frames and Ω

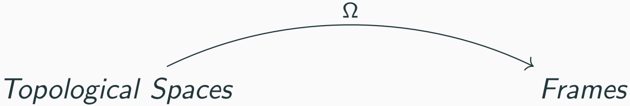
- Define $\Omega : \mathbf{X} \mapsto \tau$
- Sends each topology on a space to its "lattice of opens"
- Is this an algebraic object? Specifically: a lattice?
    - Yes, and yes!

- Since $\emptyset, \{X\} \in \tau$, we have 0 and 1
- Form the lattice-like structure under $\subseteq$ as the $\leq$ relation
    - This gives us a (bounded) partial order
- τ is closed under finite intersections and arbitrary unions Therefore, the lattice-like structure is closed under finite meets and arbitrary joins
- Hence, a lattice

From spaces to locales: Frames and Ω (cont.)

- We define a *frame* to be a *complete* lattice such that it is infinitely join-distributive: $a \wedge (\bigvee S) = \bigvee \{a \wedge s : s \in S\}$
  - Compare this with a topology being closed under arbitrary unions and finite intersections
- We will call the category of frames and frame-homomorphisms '**Frm**'

From spaces to locales: Frames and Ω (cont.)



From spaces to locales: Locales

- **Loc**
- The *dual* of **Frm**
- "homomorphisms" in **Loc** are called 'continuous maps'
    - Hints at the topological connection
- The natural corresponding place for the "lattice of opens" of spaces

From spaces to locales: Overview of structures

Complete Heyting Algebras	**Frames**	**Locales**
Heyting homomorphisms	Frame homomorphisms	Continuous maps

From spaces to locales: Ω

$$\textit{Topological Spaces} \xrightarrow{\ \Omega\ } \textit{Locales}$$

From locales to spaces

Key question: can we go back?

From locales to spaces: points

- To go from a locale to a space, we need *points* again
- Since *(topological) spaces have points*

From locales to spaces: points (cont.)

- Define a *point* in a locale, $\mathbf{A}$, as a (continuous) map: $\mathbf{2} \longrightarrow \mathbf{A}$
- Easier to see as a frame homomorphism
- $\mathbf{A} \longrightarrow \mathbf{2}$
- Compare with Boolean algebras!

From locales to spaces: points (cont.)

Another motivation

Points in **Set**

$$\{*\} \xrightarrow{\quad\quad\quad\quad x \quad\quad\quad\quad} X$$

From locales to spaces: points (cont.)

Points in **Top**

$$(\{*\}, \tau_0) \xrightarrow{\quad x \quad} (X, \tau)$$

(*Where* τ_0 *is the trivial topology*)

From locales to spaces: points (cont.)

- Always: $\Omega((\{*\}, \tau_0)) = \mathbf{2}$
- Let $\Omega((X, \tau)) = \mathbf{A} \in \mathbf{Loc}$.

From locales to spaces: points (cont.)

Points in **Loc**

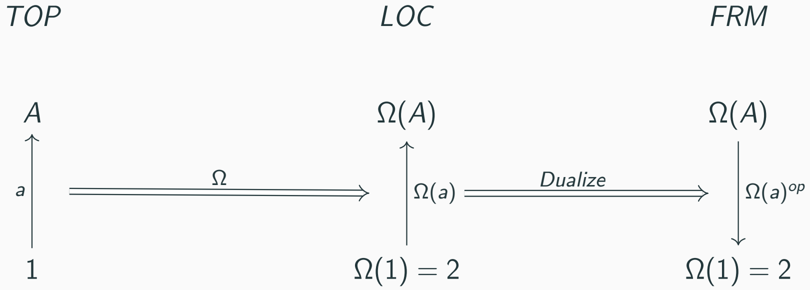
$$\Omega(\{*\}, \tau_0) = \mathbf{2} \xrightarrow{\quad a \quad} \mathbf{A}$$

From locales to spaces: points (cont.)

Points in **Frm**

$$\mathbf{A} \xrightarrow{\quad a \quad} \mathbf{2}$$

Points (cont.)



From locales to spaces: Points (cont.)

What are these points, more precisely?

- Points = Homomorphisms
- Will often look from the ”Frame perspective”
- Each point corresponds to a choice of *principal prime ideals*
  - Downsets
- Equivalently, via *completely prime filters*
  - Upsets

Upshot: we have points in our locales!

From locales to spaces: The base set

- $(X, \tau) \in$ **Top**
- What is X?
  - A set of *points*
- We want to send a locale to its *set of points*

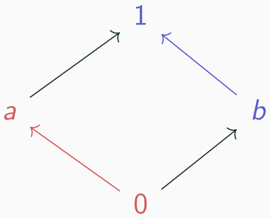
From locales to spaces: The *pt* map

- Define $pt : \mathbf{A} \mapsto pt(\mathbf{A})$ where $pt(\mathbf{A})$ is the set of points of $\mathbf{A}$.

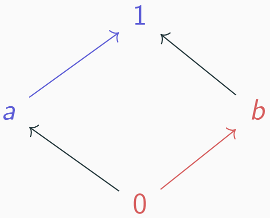
Let us look at an example

From locales to spaces: The *pt* map (cont.)

p_1



p_2



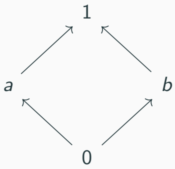
$$b, 1 \xmapsto{p_1} 1 \qquad a, 1 \xmapsto{p_2} 1$$

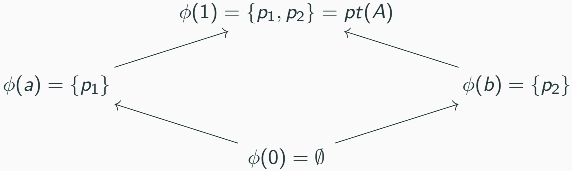
$$0, a \xmapsto{p_1} 0 \qquad 0, b \xmapsto{p_2} 0$$

Set of points $= \{p_1, p_2\}$

From locales to spaces: Getting the topology

- We now have half of the space $\mathbf{X} = (pt(\mathbf{A}), \tau)$
- How do we get τ?
- Define $\phi : \mathbf{A} \longrightarrow \mathcal{P}(pt(\mathbf{A}))$
  - via $a \mapsto \{p : \mathbf{A} \longrightarrow \mathbf{2} | p(a) = 1\}$


1
a
b
0


$\phi(1) = \{p_1, p_2\} = pt(A)$
$\phi(a) = \{p_1\}$
$\phi(b) = \{p_2\}$
$\phi(0) = \emptyset$

From locales to spaces: The whole topology

- Is $\phi(\mathbf{A})$ a topology?
  - $pt(\mathbf{A}) \in \phi(\mathbf{A})$
  - $\emptyset \in \phi(\mathbf{A})$
  - Left to prove: Closed under finite intersections and arbitrary unions
  - We will use the fact that the image of ϕ is part of a power-set algebra: $\cup$ and $\cap$ are $\vee$ and $\wedge$

From locales to spaces: The whole topology (cont.)

Proof

Want to show: $\phi(\bigvee S) = \bigcup\{\phi(a) : a \in S\}$ for $S \subseteq \mathbf{A}$

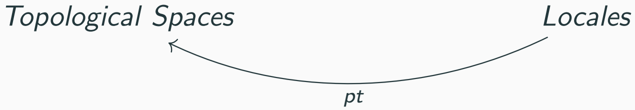
$$
\begin{aligned}
\text{Let} \quad & p \in \bigcup\{\phi(a) : a \in S\} \\
\Leftrightarrow \quad & \exists a \in S(p(a) = 1) \\
\Leftrightarrow \quad & \bigvee\{p(a) : a \in S\} = 1 \\
\Leftrightarrow \text{ (since } p \text{ homomorphism)} \quad & p(\bigvee S) = 1 \\
\Leftrightarrow \quad & p \in \phi(\bigvee S)
\end{aligned}
$$

(*Johnstone 1982*)

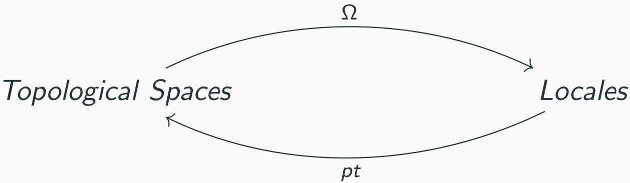
We obtain the topology $(pt(\mathbf{A}), \phi(\mathbf{A}))$

From locales to spaces: The other half of the bridge

We will denote the map $\mathbf{A} \mapsto (pt(\mathbf{A}), \phi(\mathbf{A}))$ by '*pt*' (following Johnstone 1982)



The full bridge

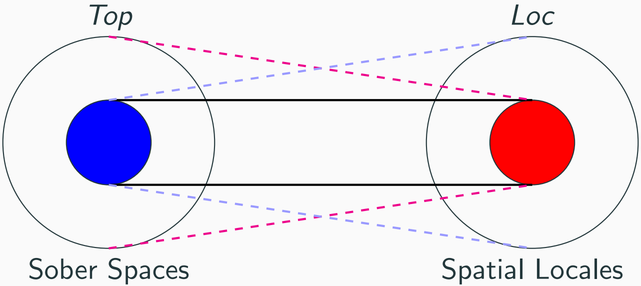


The core of the bridge

Question: Is this an equivalence?

No. Each of the maps are non-surjective on the corresponding category

The core of the bridge (cont.)



Spatial locales: Definition

- Spatial locales have *enough points*
- That is, they can separate any two non-related elements of the locale via a point-map

Formally:

$$\forall x, y \in \mathbf{A}(x \nleq y \to \exists p(p(x) = 1 \wedge p(y) = 0))$$

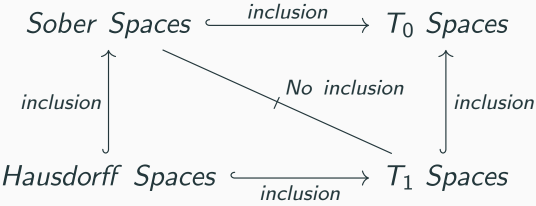
Spatial locales: A non-example

- Boolean algebras are locales (*as structures*)
- An *atom* in a lattice is an element x such that $x \neq 0$ and there is no element y such that $0 < y < x$
  - minimal non-zero element
- An atomless but complete Boolean algebra is non-spatial

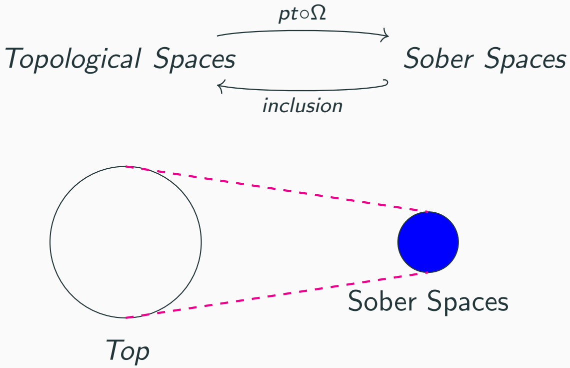
Sober spaces: Definition

- A closed set is *irreducible* if it is not the union of two proper subsets that are closed
- A topological space is *sober* if every irreducible closed set is the closure of a single (unique) point

Sober spaces: Separation axioms

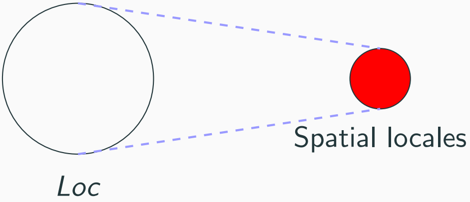


Soberfication



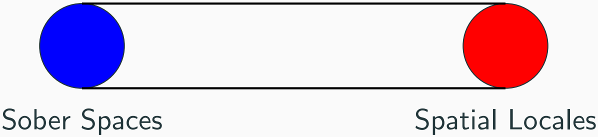
"Spatio-fication"

$$\mathit{Locales} \underset{\mathit{inclusion}}{\overset{\Omega \circ pt}{\rightleftarrows}} \mathit{Spatial\ Locales}$$



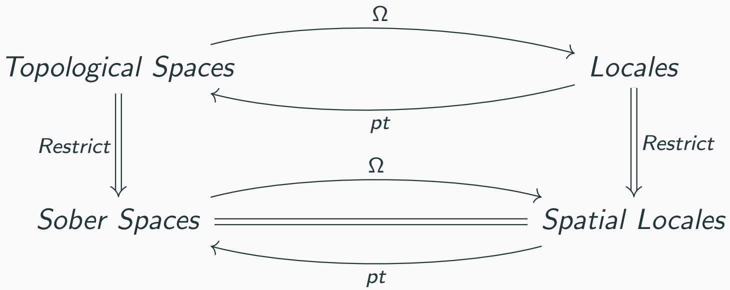
The core of the bridge: Part 2

Spatial locales $\cong$ Sober spaces



Idea: composition gives isomorphisms via soberfication/spatio-fication

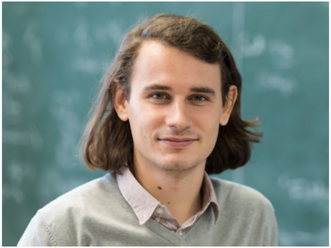
The full picture: So far



Next steps: By increasing order of insanity

- The **BRT**
- More dualities
  - Heyting algebras: Esakia spaces
  - Distributive lattives: Spectral/Coherent spaces
- Nice topology
  - Constructive proofs in topology
  - Better behaved constructions
- Stone-type dualities
  - Generalise the construction via categorical tools
- Topological algebras
  - Stone space + Boolean algebra = dual of Sets
- Topos theory
  - Locales: natural place for topos theory
  - "The real study of topology" (Grothendieck, via McLarty)

A final insanity: Condensed mathematics



Living legend Peter Scholze (*via Quanta Magazine)*

- Massive project to unify large areas of mathematics
- Replace topological spaces by *condensed sets*
- Condensed set = "blowing up" a *profinite set*
    - (via "*sheaves*" on a *site*)
- What is a profinite set?

Profinite sets = **Stone spaces**

Fin