

Topology in Graph Theory

Topology Project

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Kneser graph $K(n, k)$

- Vertices: k -subsets of $\{1, \dots, n\}$
- Edge between vertex A and B if $A \cap B = \emptyset$

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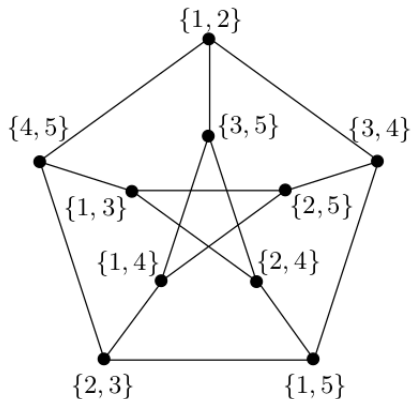
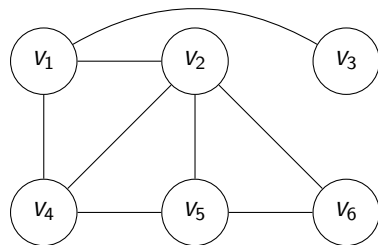


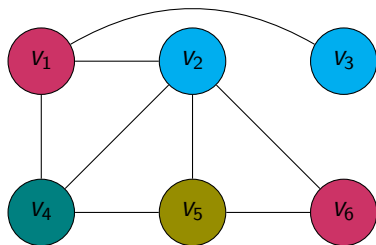
Figure: Kneser graph $K(5, 2)$, credits: Proofs from the BOOK, p. 251

Graph coloring

- No two connected vertices are colored with same color
- $\chi(G)$: minimum number of colors needed for coloring of G



(a) Graph G



(b) 4-coloured graph G

Figure: 4-colouring of a graph

3-coloring of Kneser graph $K(5, 2)$

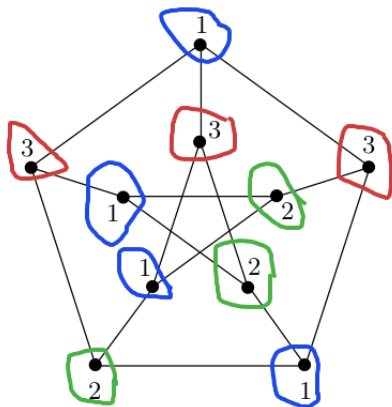


Figure: Coloring of $K(5, 2)$, credits: Proofs from the BOOK, p. 251

- Reformulation of graph coloring of a Kneser graph $K(n, k)$:
partition of vertices $V(n, k)$ in $V_1 \dots V_{d+2}$ and each V_i is not connected

Kneser conjecture

Possible coloring: Take $n := 2k + d, d \geq 0$

For $i \in \{1, \dots, d + 1\}$ with V_i containing all k -sets that have i as smallest element.
The remaining k -sets are contained in the set $\{d + 2, d + 3, \dots, 2k + d\}$, which has only $2k - 1$ elements

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But is this sufficient?

Theorem (Kneser conjecture [AZ98])

$$(K(2k + d, k)) = d + 2$$

Topology to the Rescue

- combinatorial problem

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- (23 year later) László Lovász [Lov78]: topology ,
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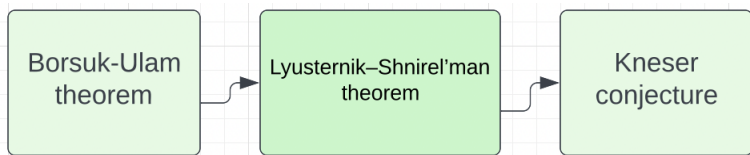


Figure: Outline of the Proofs to Come

Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam theorem [Bor33])

For every continuous map $f : S^d \rightarrow \mathbb{R}^d$ from d -sphere to d -space, there are antipodal points $x, -x$ that are mapped to the same point $f(x) = f(-x)$.

Where a d -sphere is defined as a d -dimensional object in $d + 1$ -dimensional space:
 $S^d := \{x \in \mathbb{R}^{d+1} : |x| = 1\}$

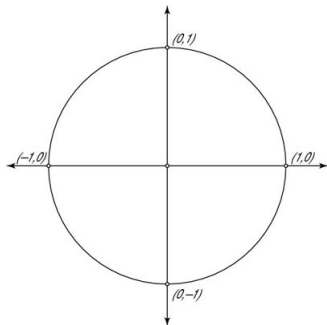


Figure: Unit Circle S^1

Proof Borsuk-Ulam ($d = 1$)

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Proof

Given a continuous function $f : S^1 \rightarrow \mathbb{R}$, we define $g : S^1 \rightarrow \mathbb{R}$ as follows:

$$g(x) := f(x) - f(-x)$$

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Observe that $g(-x) = f(-x) - f(x) = -(f(x) - f(-x)) = -g(x)$ and g continuous because sum of continuous functions.

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Observe that $g(-x) = f(-x) - f(x) = -(f(x) - f(-x)) = -g(x)$ and g continuous because sum of continuous functions. Wlog assume $g(x) > 0$ and thus $g(-x) < 0$. As g continuous on S^1 from $-x$ to x , and $g(-x) < 0 < g(x)$ by the Intermediate Value Theorem, there exists x on S^1 s.t. $g(x) = 0$.

Borsuk-Ulam in General

- Borsuk-Ulam holds for any integer d
- $d = 2$: unit sphere (our world) to $2d$ -plane

Figure: " S^2 sphere"

- antipodal points (opposite sides) where pressure and temperature are exact same

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- antipodal points (opposite sides) where pressure and temperature are exact same
- One more step to use this topological result for our graph theory problem

Proof of Lyusternik–Shnirel'man theorem (1)

Theorem (Lyusternik–Shnirel'man theorem [L L30])

If the d -sphere S^d is covered by $d + 1$ sets,

$$S^d = U_1 \cup \dots \cup U_d \cup U_{d+1},$$

such that each of the first d sets U_1, \dots, U_d is sphered $+ 1$ sets contains a pair of antipodal points $x, -x$.

Proof of Lyusternik{Shnirel'man theorem (1)

If the d -sphere S^d is covered by $d+1$ sets,

$$S^d = U_1 \cup \dots \cup U_d \cup U_{d+1};$$

such that each of the first d sets U_1, \dots, U_d is spherically antipodal-free, then the $(d+1)$ sets contains a pair of antipodal points $x, -x$.

Assume for contradiction that none of the U_i 's contain antipodal points. Define $f : S^d \rightarrow \mathbb{R}^d$ as

$$f(x) := (\text{dist}(x; U_1); \text{dist}(x; U_2); \dots; \text{dist}(x; U_d))$$

Proof of Lyusternik{Shnirel'man theorem (1)

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such that each of the first d sets U_1, \dots, U_d is spherically antipodal-free, then the last set U_{d+1} contains a pair of antipodal points $x, -x$.

Assume for contradiction that none of the U_i 's contain antipodal points. Define $f : S^d \rightarrow \mathbb{R}^d$ as

$$f(x) := (\text{dist}(x; U_1); \text{dist}(x; U_2); \dots; \text{dist}(x; U_d))$$

As f is continuous, Borsuk-Ulam yields existence of antipodal x s.t. $f(x) = f(-x)$.

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$$S^d = U_1 \cup \dots \cup U_d \cup U_{d+1};$$

such that each of the first d sets U_1, \dots, U_d is spherically antipodal-free, i.e. no set contains a pair of antipodal points $x, -x$.

Assume for contradiction that none of the U_i 's contain antipodal points. Define $f : S^d \rightarrow \mathbb{R}^d$ as

$$f(x) := (\text{dist}(x; U_1); \text{dist}(x; U_2); \dots; \text{dist}(x; U_d))$$

As f is continuous, Borsuk-Ulam yields existence of antipodal x s.t. $f(x) = f(-x)$. As U_{d+1} does not contain both x and $-x$, wlog, there is some U_k ($1 \leq k \leq d$) s.t. $x \in U_k$.

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Proof

Assume for contradiction that none of the U_i 's contain antipodal points. Define $f : S^d \rightarrow \mathbb{R}^d$ as

$$f(x) := (\text{dist}(x, U_1), \text{dist}(x, U_2), \dots, \text{dist}(x, U_d))$$

As f continuous, Borsuk-Ulam yields existence antipodal $x, -x$ s.t. $f(x) = f(-x)$. As U_{d+1} does not contain both x and $-x$, wlog, there is some U_k ($1 \leq k \leq d$) s.t. $x \in U_k$. This means $\text{dist}(x, U_k) = 0$, and thus $\text{dist}(-x, U_k) = 0$. (continued on next slide)

Proof of the Lyusternik–Shnirel'man theorem (2)

Proof

Assume for contradiction that none of the U_i 's contain antipodal points. Define $f : S^d \rightarrow \mathbb{R}^d$ as

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If U_k is closed, then $-x \in U_k$.

Proof of the Lyusternik–Shnirel'man theorem (2)

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If U_k is closed, then $-x \in U_k$.

If U_k is open, then $-x \in \bar{U}_k$. We have $\bar{U}_k \subset S^d \setminus (-U_k)$ because closed subset of S^d that contains U_k . This means $-x \notin -U_k$. But then, $x \notin U_k$.

Proof of the Kneser conjecture (1)

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Proof

- Take $2k + d$ points such that no $d + 2$ lay on a hyperplane through the center of S^{d+1}
- Assume partition $V_1 \dots V_{d+1}$

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Proof

- Take $2k + d$ points such that no $d + 2$ lay on a hyperplane through the center of S^{d+1}

- Assume partition $V_1 \dots V_{d+1}$

- Define:

$O_i := \{x \in S^{d+1} \mid \text{the open hemisphere } H_x \text{ with pole } x \text{ contains a } k\text{-set from } V_i\}$

$C := S^{d+1} \setminus (O_1 \cup \dots \cup O_{d+1})$
all O_i with C cover S^{d+1}

Figure: Credits: Proofs from the BOOK, p. 251

Proof of the Kneser conjecture (2)

Lyusternik-Shnirel'man

) antipodal points x ; x in O_i for one i or in C

Proof of the Kneser conjecture (2)

Lyusternik-Shnirel'man

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Case 1: x ; $x \in C$

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Case 1: x ; $x \in C$

) hemispheres H_x ; H_x contain less than k points

Proof of the Kneser conjecture (2)

Lyusternik-Shnirel'man

) antipodal points x, x in O_i for one i or in C

Case 1: $x, x \in C$

) hemispheres H_x, H_x contain less than k points

) at least $d + 2$ points on equator

Proof of the Kneser conjecture (2)

Proof

- Lyusternik–Shnirel'man
antipodal points $x, -x$ in O_i for one i or in C

Case 1: $x, -x \in C$

hemispheres H_x, H_{-x} contain less than k points
at least $d + 2$ points on equator
but we chose them not to be

Proof of the Kneser conjecture (3)

Proof

Case 2: $x, -x \in O_i$

Proof of the Kneser conjecture (3)

Proof

Case 2: $x, -x \in O_i$

- exists k -sets $A, B \subseteq V_i$ with $A \cap H_x$ and $B \cap H_{-x}$

Proof of the Kneser conjecture (3)

Case 2: $x \in V_i$

exists k -sets $A, B \subseteq V_i$ with $A \cap B = \emptyset$ and $|A| = |B| = k$

-) A, B disjoint
-) exists edge from A to B

Proof of the Kneser conjecture (3)

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exists k -sets $A, B \subseteq V_i$ with $A \cap B = \emptyset$ and $|A| = |B| = k$

-) A, B disjoint
-) exists edge from A to B
-) Contradiction! So $K(2k + d; k)$ cannot equal $d + 1 \dots$

Figure: Credits: Proofs from the BOOK, p. 251

Wrapping Up

Topology useful in unexpected places

Beginnings of Topological Graph Theory

Matousek [Mat04] also proved the conjecture via a combinatorial argument

Sources I

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THANKS FOR LISTENING!

(THE END)

QUESTIONS?