Topology in Graph Theory

Topology Project

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Kneser graph $K(n, k)$

- Vertices: k-subsets of $\{1, \ldots, n\}$
- Edge between vertex A and B if $A \cap B = \emptyset$

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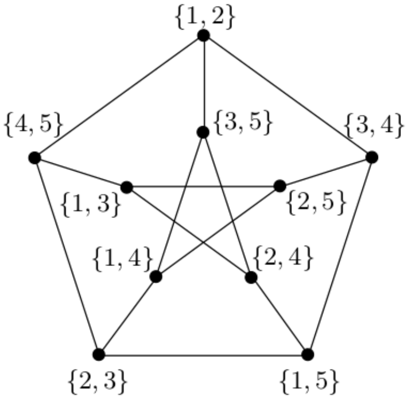
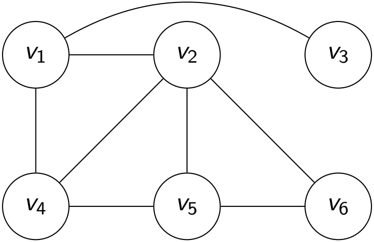


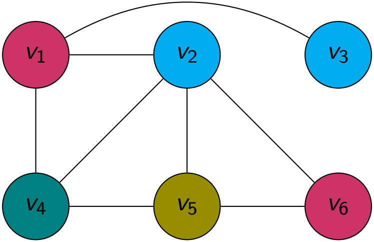
Figure: Kneser graph $K(5, 2)$, credits: Proofs from the BOOK, p. 251

Graph coloring

- No two connected vertices are colored with same color
- $\chi(G)$: miminum number of colors needed for coloring of G



(a) Graph G



(b) 4-coloured graph G

Figure: 4-colouring of a graph

3-coloring of Kneser graph $K(5,2)$

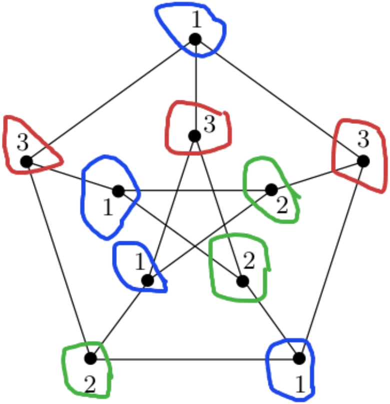


Figure: Coloring of $K(5,2)$, credits: Proofs from the BOOK, p. 251

- Reformulation of graph coloring of a Kneser graph $K(n,k)$:

$\Rightarrow$ partition of vertices $V(n,k)$ in $V_1 \cup \ldots \cup V_{d+2}$ and each V_i is not connected

Kneser conjecture

Possible coloring: Take $n := 2k + d, d \geq 0$
For $i \in \{1, ..., d+1\}$ with V_i containing all k-sets that have i as smallest element.
The remaining k-sets are contained in the set $\{d+2, d+3, ..., 2k+d\}$, which has only $2k-1$ elements

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But is this sufficient?

Theorem (Kneser conjecture [AZ98])

$$\chi(K(2k+d, k)) = d+2$$

Topology to the Rescue

- combinatorial problem

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- (23 year later) László Lovász [Lov78]: topology ☺
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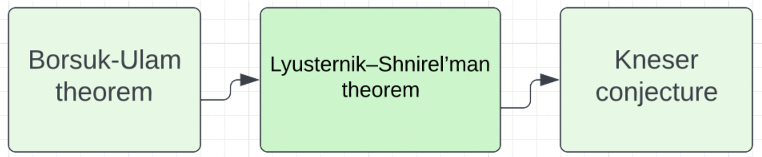

Figure: Outline of the Proofs to Come

Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam theorem [Bor33])

For every continuous map $f : S^d \to \mathbb{R}^d$ from d-sphere to d-space, there are antipodal points $x^*, -x^*$ that are mapped to the same point $f(x^*) = f(-x^*)$.

Where a d-sphere is defined as a d-dimensional object in $d+1$-dimensional space: $S^d := \{x \in \mathbb{R}^{d+1} : |x| = 1\}$

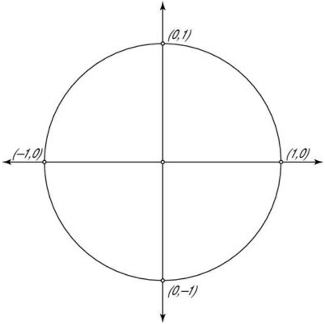

Figure: Unit Circle S^1

Proof Borsuk-Ulam ($d = 1$)

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Proof

Given a continuous function $f : S^1 \to \mathbb{R}$, we define $g : S^1 \to \mathbb{R}$ as follows:

$$g(x) := f(x) - f(-x)$$

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Borsuk-Ulam in General

- Borsuk-Ulam holds for any integer d
- $d = 2$: unit sphere (our world) to $2d$-plane

Figure: "S^2 sphere"

- antipodal points (opposite sides) where pressure and temperature are exact same

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Figure: "S^2 sphere"

- antipodal points (opposite sides) where pressure and temperature are exact same
- One more step to use this topological result for our graph theory problem

Proof of Lyusternik–Shnirel'man theorem (1)

Theorem (Lyusternik–Shnirel'man theorem [L L30])

If the d-sphere S^d is covered by $d+1$ sets,

$$S^d = U_1 \cup ... \cup U_d \cup U_{d+1},$$

such that each of the first d sets $U_1, ..., U_d$ is sphered $+1$ sets contains a pair of antipodal points $x^*, -x^*$.

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Assume for contradiction that none of the U_i's contain antipodal points. Define $f : S^d \to \mathbb{R}^d$ as

$$f(x) := (dist(x, U_1), dist(x, U_2), ..., dist(x, U_d))$$

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Proof of the Lyusternik–Shnirel'man theorem (2)

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If U_k is closed, then $-x^* \in U_k$. ↯

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If U_k is closed, then $-x^* \in U_k$. ↯
If U_k is open, then $-x^* \in \bar{U}_k$. We have $\bar{U}_k \subseteq S^d \setminus (-U_k)$ because closed subset of S^d that contains U_k. This means $-x^* \notin -U_k$. But then, $x^* \notin U_k$. ↯ $\square$

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- Assume partition $V_1 \cup \ldots \cup V_{d+1}$
- Define:

$O_i := \{x \in S^{d+1} \mid \text{the open hemisphere } H_x \text{ with pole } x \text{ contains a } k\text{-set from } V_i\}$
$C := S^{d+1} \setminus (O_1 \cup \ldots \cup O_{d+1})$
$\Rightarrow$ all O_i with C cover S^{d+1}

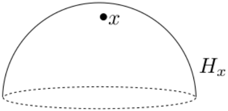

Figure: Credits: Proofs from the BOOK, p. 251

Proof of the Kneser conjecture (2)

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- Lyusternik–Shnirel'man

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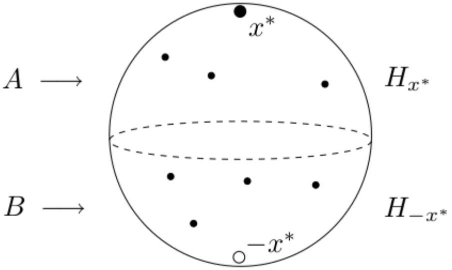
$\lightning$ but we chose them not to be



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$\Rightarrow$ A, B disjoint

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$\Rightarrow$ Contradiction! So $\chi(K(2k+d,k))$ cannot equal $d+1$... $\square$

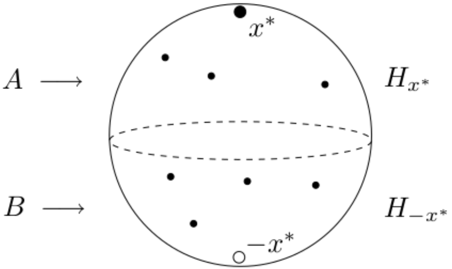

Figure: Credits: Proofs from the BOOK, p. 251

Wrapping Up

- Topology useful in unexpected places
- Beginnings of Topological Graph Theory
- Matousek [Mat04] also proved the conjecture via a combinatorial argument

[AZ98] Martin Aigner and Günter M. Ziegler. *Proofs from the BOOK*. International series of monographs on physics. https://doi.org/10.1007/978-3-662-57265-8. Springer Berlin, Heidelberg, 1998, pp. 251–255.

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[Mat04] Jiří Matouek. "A Combinatorial Proof of Kneser's Conjecture*". In: *Combinatorica* 24 (2004), pp. 163–170.

$\mathbb{THANKS\ FOR\ LISTENING}$!

$(\Leftrightarrow \mathbb{THE\ END})$

$\Rightarrow \mathbb{QUESTIONS}$?